End of time in vector field theories

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Summary

- Self interaction: Anything beyond the kinetic and mass terms. Example: Higgs.
- Main message: Self-interacting vector-fields are not physical in the strict sense
- Time evolution is possible, but not indefinitely: Problem is not detectable by "local" methods.
- Related to the "conditions" on the vectors: Likely goes beyond vectors.

Coates and Ramazanoglu, PRL 2022; arXiv:2211.08027 [gr-qc]
Closely related: Clough et al, PRL 2022; Mou and Zhang, PRL 2022
Inspiration: Silva et al PRD 2022; Demirboga et al PRD 2022; Garcia-Saenz et al PRL 2021
Earlier work: Esposito-Fareése et al PRD 2010

Massless scalars

$$-\partial_t^2\phi + \partial_x^2\phi = 0 \tag{1}$$

$$\phi = e^{-i\omega t}e^{ikx} \;\Rightarrow\; \omega = \pm k \;\rightarrow\; \phi = e^{ikx}e^{\pm ikt}$$

.
Fourier modes oscillate in time.

Massive scalars

$$-\partial_t^2\phi + \partial_x^2\phi = m^2\phi \tag{2}$$

$$\phi = e^{-i\omega t}e^{ikx} \;\Rightarrow\; \omega = \pm\sqrt{k^2+m^2} \;\rightarrow\; \phi = e^{ikx}e^{\pm i\sqrt{k^2+m^2}t}$$

.

Fourier modes *still* oscillate in time.

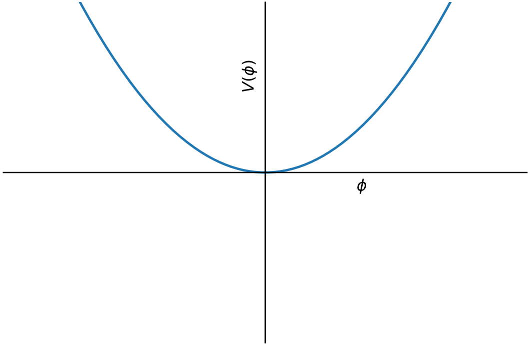
Massive scalars: alternative look

$$-\partial_t^2\phi + \partial_x^2\phi = V'(\phi)\ ,\ \ V(\phi) = \frac{1}{2}m\phi^2 \quad (3)$$

$$\partial_x\phi = 0 \ \Rightarrow \ \ddot{\phi} = -V'(\phi) \quad (4)$$

$$\Rightarrow \ \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 = const \quad (5)$$

Particle moving under the potential $V(\phi)$.



Self-interacting scalars

$$-\partial_t^2\phi + \partial_x^2\phi = V'(\phi) \tag{6}$$

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda m^2\phi^4 \tag{7}$$

$\lambda > 0 \rightarrow$ fine

$\lambda < 0 \rightarrow$ potential trouble

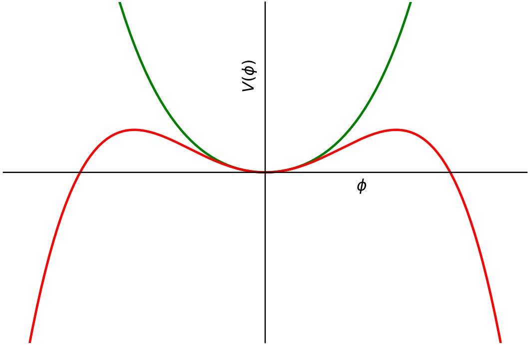


Figure: $V(\phi) = \frac{1}{2}\phi^2 \pm \frac{1}{4}\phi^4$

Changing the derivative terms – Ill-posedness

$$+\partial_t^2\phi + \partial_x^2\phi = 0 \tag{8}$$

$$\phi = e^{-i\omega t}e^{ikx} \;\Rightarrow\; \omega = \pm ik \;\rightarrow\; \phi = e^{ikx}e^{|k|t}$$

.

Arbitrarily fast blow up for $k \rightarrow \infty$!
There are modes that *immediately* go to infinity if we try time evolution.

$$\begin{aligned} &\phi(0, x=0)\;,\; \partial_t\phi(0,x) = \epsilon e^{i\frac{x}{\epsilon}} \\ \Rightarrow\, &\phi(x,t) = \epsilon^2 e^{i\frac{x}{\epsilon}}\;\sinh\frac{t}{\epsilon} \;\rightarrow\; \text{no continuous dependence!} \end{aligned} \tag{9}$$

General case

$$A\ \partial_t^2\phi + 2B\ \partial_t\partial_x\phi + C\ \partial_x^2\phi \ = \ g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \ \Rightarrow \ g^{\mu\nu} = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \tag{10}$$

Well-posed if eigenvalues are $(-,+,\dots,+)$: **Signature of the metric.**
Lower derivative terms are not critical.

Scalar field summary

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = V'(\phi) \tag{11}$$

- $V(\phi)$ unbounded from below: Bad physics, math *maybe* OK.
- Metric signature not Lorentzian: ill posed, even math is in trouble.
- The highest derivative terms determine well-posedness, the metric has to be Lorentzian. Good thing is that the metric is a given ... or is it?

Massless vectors

$$S = -\int d^n x \sqrt{|g|}\; F_{\mu\nu} F^{\mu\nu} \tag{12}$$

$$F_{\mu\nu} = \nabla_\mu X_\nu - \nabla_\nu X_\mu = \partial_\mu X_\nu - \partial_\nu X_\mu \tag{13}$$

$$0 = \overbrace{\eta^{\mu\nu}\nabla_\mu \nabla_\nu}^{-\partial_t^2 + \partial_x^2} X^\rho - \nabla^\rho \nabla_\mu X^\mu = \nabla_\mu F^{\mu\rho} \tag{14}$$

X^μ has gauge freedom! We can *choose* X^μ such that $\nabla_\mu X^\mu = \partial_t \Phi + \nabla \cdot \vec{A} = 0 \rightarrow$ Lorenz gauge.

$$0 = \eta^{\mu\nu}\nabla_\mu \nabla_\nu \; X^\rho \tag{15}$$

Massive vectors

$$S = -\int d^n x \sqrt{|g|} \ \left[F_{\mu\nu} F^{\mu\nu} + 2m^2 X_\mu X^\mu\right] \tag{16}$$

$$F_{\mu\nu} = \nabla_\mu X_\nu - \nabla_\nu X_\mu = \partial_\mu X_\nu - \partial_\nu X_\mu \tag{17}$$

$$m^2 X^\rho = \eta^{\mu\nu} \nabla_\mu \nabla_\nu X^\rho - \nabla^\nu \nabla_\mu X^\mu \tag{18}$$

$$\nabla_\rho \nabla_\mu F^{\mu\rho} = m^2 \nabla_\rho X^\rho \tag{19}$$

$$\Rightarrow \ \nabla_\mu X^\mu = 0 \rightarrow \text{ Lorenz condition, not gauge!} \tag{20}$$

Self-interacting vectors

$$S = -\int d^n x \sqrt{|g|} \; \left[F_{\mu\nu}F^{\mu\nu} + 2m^2 X_\mu X^\mu + \lambda m^2 (X_\mu X^\mu)^2\right] \tag{21}$$

$$m^2(1+\lambda X^2)X^\rho = \eta^{\mu\nu}\nabla_\mu\nabla_\nu X^\rho - \nabla^\rho\nabla_\mu X^\mu \tag{22}$$

$$\nabla_\rho\nabla_\mu F^{\mu\rho} = m^2 \; \nabla_\rho \left[(1+\lambda X^2)X^\rho\right] \tag{23}$$

$$\Rightarrow \; \nabla_\rho \left[(1+\lambda X^2)X^\rho\right] = 0 \rightarrow \; \text{No easy cancellation} \tag{24}$$

Self-interacting vectors: Principal part

$$m^2(1+\lambda X^2)X^\rho = \eta^{\mu\nu}\nabla_\mu\nabla_\nu X^\rho - \nabla^\rho\nabla_\mu X^\mu \tag{25}$$

... generalized Lorenz condition+algebra ...

$$\Rightarrow\ 0 = \left[(1+\lambda X^2)\eta^{\mu\nu} + 2\lambda X^\mu X^\nu\right]\ \nabla_\mu\nabla_\nu X^\rho + \dots \tag{26}$$

$$\Rightarrow\ 0 = \bar{g}^{\mu\nu}\ \partial^\mu\partial^\nu X^\alpha + \dots \tag{27}$$

$\bar{g}^{\mu\nu} = (1+\lambda X^2)\begin{bmatrix} -1+\frac{2\lambda X^t X^t}{1+\lambda X^2} & \frac{2\lambda X^t X^x}{1+\lambda X^2} \\ \frac{2\lambda X^t X^x}{1+\lambda X^2} & 1+\frac{2\lambda X^x X^x}{1+\lambda X^2}\end{bmatrix}$ governs the dynamics, *not* $\eta^{\mu\nu}$!

Even if the spacetime is perfectly OK, $\bar{g}_{\mu\nu}(X^\mu)$ can have the wrong signature!

Self-interacting vectors: Principal part

- Works the same way for a general curved spacetime $\eta_{\mu\nu} \to g_{\mu\nu}$.
- However, curvature plays no essential role, *intrinsic* to the vector field theory, i.e. even occurs for fixed $g_{\mu\nu} = \eta_{\mu\nu}$.
- More technical machinery is needed for higher than $1+1$D, but the same result.

Problematic $\bar{g}_{\mu\nu}$

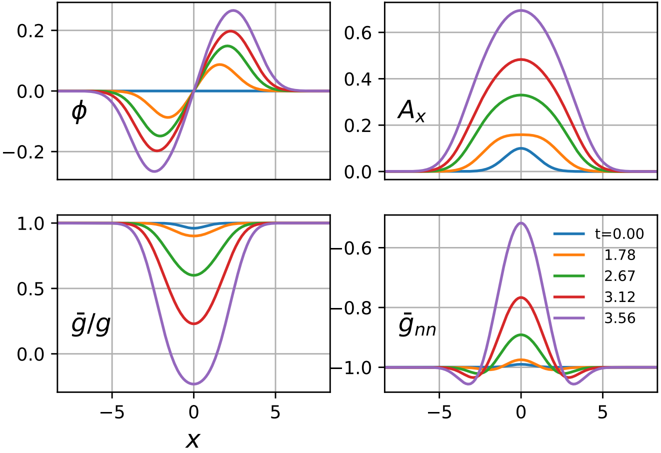
Singular metric when determinant vanishes

$$\begin{aligned} \bar{g} &= \det \bar{g}_{\mu\nu} = g \left(1 + \lambda X^2\right)^d \left(1 + 3\lambda X^2\right) = g \; z^d \; z_3 = 0 && (28) \\ \Rightarrow X^2 &= -\lambda/3 && (29) \\ \bar{g}_{\mu\nu} & \text{ becomes singular at finite values of } X^\mu \\ & \text{ confirmed by checking the divergence of curvature} \end{aligned}$$

Dynamical loss of hyperbolicity

Start with Lorentzian $\bar{g}_{\mu\nu}$, does the problem occur?

Yes, for any value of m, λ.



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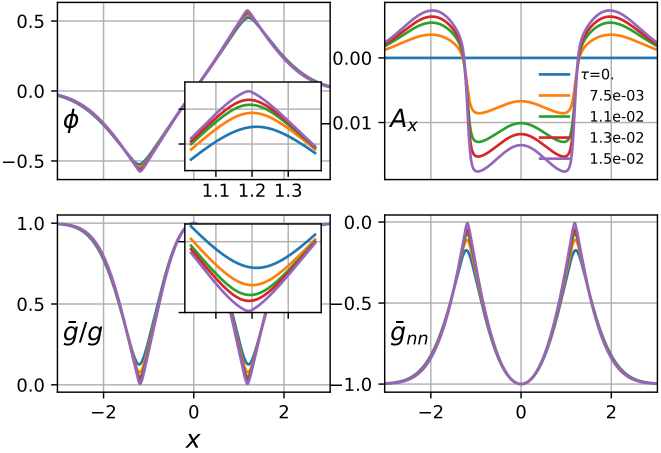

On progress, future

- Likely holds for anything beyond Proca.
- Likely holds for other theories: $p-$form fields. **New theoretical test!**
- Backreaction effects not known.
- Similar problems in nonminimal couplings, tachyon on vector $\Rightarrow$ ill-posedness.
- Might be overlooked/resolved when the theory is *effective*.

Effective field theories, fixing-the-equation

Abelian Higgs theory: define $D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi$.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left[\left(\overline{D_\mu\phi}\right)\left(D^\mu\phi\right) - m^2|\phi|^2 + \frac{\Lambda}{2}|\phi|^4\right] \tag{30}$$

$$\Rightarrow \nabla_\mu F^{\mu\nu} = -q\rho^2\partial^\nu\Theta + q^2\rho^2A^\nu \quad , \quad (\phi = \rho\,\exp(i\Theta)) \tag{31}$$

$$\Rightarrow \nabla_\mu F^{\mu\nu} = \frac{q^2m^2}{\Lambda}\left(1 - \frac{q^2X^2}{m^2}\right)X^\nu + \mathcal{O}\left(\frac{1}{m^2}\right) \tag{32}$$

The first equation is fine, no loss of hyperbolicity.
Only known for $\lambda < 0$, other UV completion options?

The end

QUESTIONS?

Some wrong turns

Evolve the vector with $3+1$ decomposition.

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{33}$$

$$X_\mu = n_\mu \phi + A_\mu \ , \ \phi = -n_\mu X^\mu \ , \ A_i = (\delta^\mu{}_i + n^\mu n_i)\, X_\mu \ .$$

$$\begin{aligned} \partial_t \phi &= \beta^i D_i \phi - A^i D_i \alpha - \frac{\alpha}{\bar{g}_{nn}} z \left(K\phi - D_i A^i\right) \\ &+ \frac{2\lambda\alpha}{\bar{g}_{nn}} \left[A^i A^j D_i A_j - \phi\left(E_i A^i - K_{ij} A^i A^j + 2A^i D_i \phi\right)\right] \\ 0 &= D_i E^i + \mu^2 z \phi = \mathcal{C} \ , \end{aligned} \tag{34}$$

$$\bar{g}_{nn} = n^\mu n^\nu \bar{g}_{\mu\nu} = -(1 + 3\lambda X^2) + 2\lambda A_i A^i \ . \tag{35}$$

A numerical look at ill-posedness

$$+\partial_t^2 \phi + \partial_x^2 \phi = 0 \tag{36}$$

$$\phi = e^{-i\omega t} e^{ikx} \;\Rightarrow\; \omega = \pm ik \;\rightarrow\; \phi = e^{ikx} e^{|k| t}$$

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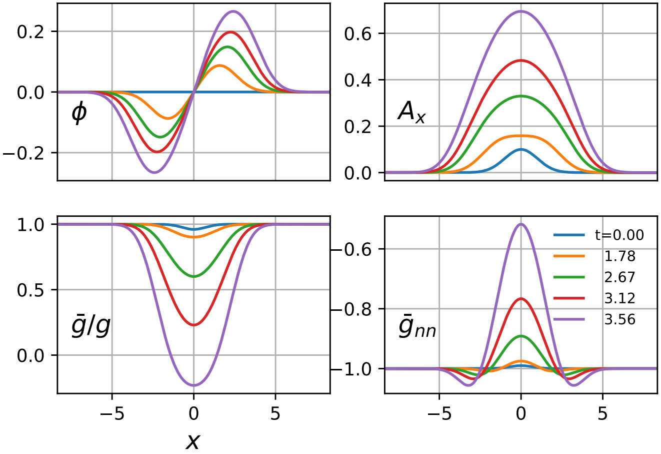

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