# Buchdahl-inspired spacetimes & wormholes

## Breaking a six-decades-old impasse

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# Buchdahl's legacies

1993 C.E.K. Mees Medal

Hans A Buchdahl *

For contributions to physical and geometrical optics, their significance, and their international import

## Buchdahl's theorem

Article Talk

From Wikipedia, the free encyclopedia

In general relativity, **Buchdahl's theorem**, named after Hans Adolf Buchdahl,[1] makes more precise the notion that there is a maximal sustainable density for ordinary gravitating matter. It gives an inequality between the mass and radius that must be satisfied for static, spherically symmetric matter configurations under certain conditions. In particular, for areal radius $R$, the mass $M$ must satisfy

$$M < \frac{4Rc^2}{9G}$$

where $G$ is the gravitational constant and $c$ is the speed of light. This inequality is often referred to as **Buchdahl's bound**. The bound has historically also been called Schwarzschild's limit as it was first noted by Karl Schwarzschild to exist in the special case of a constant density fluid.[2] However, this terminology should not be confused with the Schwarzschild radius which is notably smaller than the radius at the Buchdahl bound.

# Hans Buchdahl = Father of $f(\mathcal{R})$ gravity

*Mon. Not. R. astr. Soc.* (1970), **150**, 1–8.

NON-LINEAR LAGRANGIANS AND COSMOLOGICAL THEORY

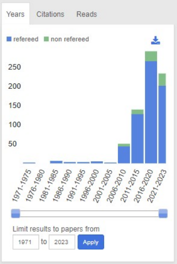
*H. A. Buchdahl*

(Communicated by W. H. McCrea)

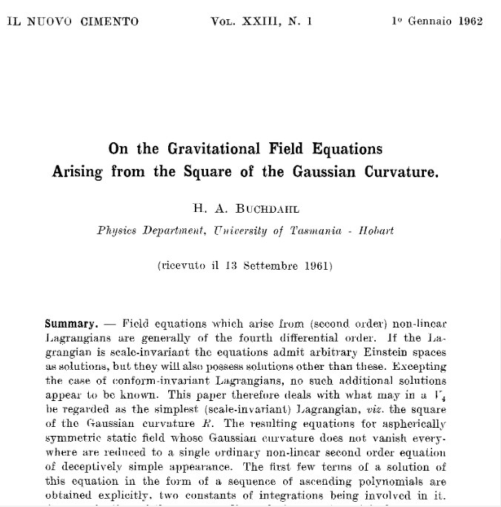
(Received 1970 May 13)

SUMMARY

In relativistic cosmology the theory of uniform model universes is based on Einstein's equations, which derive from a variational principle the field-Lagrangian $L$ of which is the scalar curvature $R$ to within an arbitrary additive constant. In this work the possibility of taking $L$ to be a more general invariant of the Riemann tensor is contemplated. The consequences of choosing $L$ to be a function $\phi$ of $R$ alone are tentatively examined under specialized circumstances, with particular attention to an open world-model oscillating between non-singular states. Difficulties revolving about the actual form which $\phi$ might take are discussed.



# Lesser known fact: Pioneer of $\mathcal{R}^2$ gravity



IL NUOVO CIMENTO Vol. XXIII, N. 1 1° Gennaio 1962

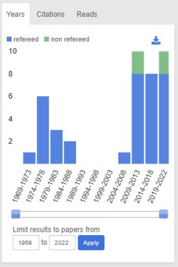
**On the Gravitational Field Equations Arising from the Square of the Gaussian Curvature.**

H. A. Buchdahl

*Physics Department, University of Tasmania - Hobart*

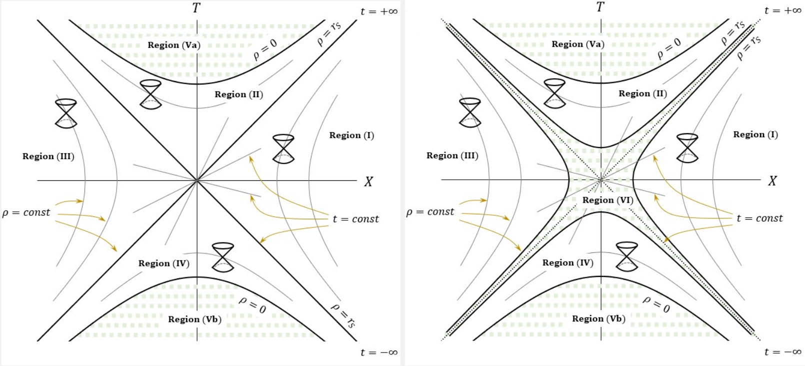
(ricevuto il 13 Settembre 1961)

**Summary.** — Field equations which arise from (second order) non-linear Lagrangians are generally of the fourth differential order. If the Lagrangian is scale-invariant the equations admit arbitrary Einstein spaces as solutions, but they will also possess solutions other than these. Excepting the case of conform-invariant Lagrangians, no such additional solutions appear to be known. This paper therefore deals with what may in a $V_4$ be regarded as the simplest (scale-invariant) Lagrangian, *viz.* the square of the Gaussian curvature $K$. The resulting equations for aspherically symmetric static field whose Gaussian curvature does not vanish everywhere are reduced to a single ordinary non-linear second order equation of deceptively simple appearance. The first few terms of a solution of this equation in the form of a sequence of ascending polynomials are obtained explicitly, two constants of integrations being involved in it.



# Buchdahl's hidden treasure trove...

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + r^2 d\Omega^2$$




**Buchdahl-inspired spacetimes !**

Presentation based on:

- *Beyond Schwarzschild-de Sitter spacetimes I*, Phys. Rev. D 106, 104004 (2022), 2211.01769 [gr-qc]
- *Beyond Schwarzschild-de Sitter spacetimes II*, 2211.03542 [gr-qc]; 2nd round review at PhysRevD
- *Beyond Schwarzschild-de Sitter spacetimes III*, 2211.07380 [gr-qc]; 2nd round review at PhysRevD
- *Traversable Buchdahl wormholes in quadratic gravity* (draft available)
- *Newtonian potential in pure $\mathcal{R}^2$ gravity on a de Sitter background* (draft available)

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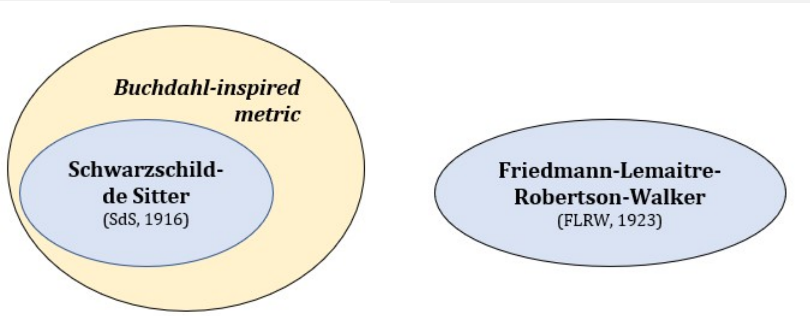
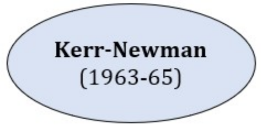
My thanks to:

- Richard Shurtleff
- Dieter Lüst
- Tiberiu Harko
- Mustapha Azreg-Aïnou
- Timothy Clifton
- PRD anonymous referees

# The Triad of Flagship Metrics




Any viable enlargement must pass tests in Solar System & binary stars !

# The basics of $\mathcal{R}^2$ gravity

$$\frac{1}{16\pi G}\mathcal{R} + \mathcal{L}_m \quad \text{vs} \quad \frac{1}{2\kappa}\mathcal{R}^2 + \mathcal{L}_m$$

- Parsimonious

$$\cancel{\frac{1}{16\pi G}\mathcal{R} + \beta\mathcal{R}^2} + \mathcal{L}_m$$

- Scale invariant
  - $\kappa$ is dimensionless ( whereas $G$ is dimensionful ! )
  - Must build $G$ from $\kappa$. Will show you how !
- Helps with renormalizability
  UV behavior is improved thanks to $1/k^4$ [Stelle-1977]
- One extra degree of freedom ...
  - In Einstein frame

$$\tilde{\mathcal{R}} + \left(\tilde{\nabla}\phi\right)^2 \tag{1}$$

  - Free of ghost (Ostrogradsky theorem [Woodard-2015])

GR theory – Einstein field eqn:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 0 \tag{2}$$

has vacuum solutions, Ricci flat: $\mathcal{R}_{\mu\nu} = 0$

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$\mathcal{R}^2$ field eqn:

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \mathcal{R}\right) + \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) \mathcal{R} = 0 \tag{3}$$

$$\Rightarrow \quad \text{Trace:} \quad \Box\,\mathcal{R} = 0 \tag{4}$$

automatically has vacuo solutions of constant $\mathcal{R}$:

(i) (trivial) Ricci-scalar-flat spaces $\mathcal{R} = 0$

(ii) Einstein spaces, $\mathcal{R}_{\mu\nu} = \Lambda g_{\mu\nu}$, with $\mathcal{R} = 4\Lambda$

Surprise #1: I shall reveal solutions of non-constant $\mathcal{R}$ !

# Does Newtonian limit exist in pure $\mathcal{R}^2$ gravity ?

Curiously, an overwhelming body of preceding works conclude:
*"No Newtonian limit in pure $\mathcal{R}^2$ gravity. One must consider $\mathcal{R}+\mathcal{R}^2$ ... "*

**There is a major flaw, however ...**

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Fortschritte der Physik
Progress of Physics

Original Paper

## Aspects of quadratic gravity

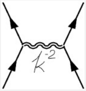
*Luis Alvarez-Gaume[1,*], Alex Kehagias[2,3], Costas Kounnas[4], Dieter Lüst[1,5,6], and Antonio Riotto[3]*

Received 13 December 2015, accepted 13 December 2015
Published online 27 January 2016

We discuss quadratic gravity where terms quadratic in the curvature tensor are included in the action. After reviewing the corresponding field equations, we analyze in detail the physical propagating modes in some specific backgrounds. First we confirm that the pure $R^2$ theory is indeed ghost free. Then we point out that for flat backgrounds the pure $R^2$ theory propagates only a scalar massless mode and no spin-two tensor mode. However, the latter emerges either by expanding the theory around curved backgrounds like de Sitter or anti-de Sitter, or by changing the long-distance dynamics by introducing the standard Einstein term. In both cases, the theory is modified in the infrared and a propagating graviton is recovered. Hence we

## Alvarez-Gaume et al, "Aspects of quadratic gravity" [AlvarezGaume-2015]

Pure $\mathcal{R}^2$ on **de Sitter** background propagates **massless** spin-2 graviton despite the **absence** of Einstein-Hilbert term




$$\Rightarrow \quad V(\vec{r}) = (-i)^2 \int \frac{d^3k}{(2\pi)^2} \frac{e^{i\vec{k}.\vec{r}}}{k^2} = -\frac{1}{4\pi\, r} \quad \text{Newtonian tail !}$$

Can we prove this classically? Let's couple grav field with normal matter ...

$$\mathcal{R}\Big(\mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu}\mathcal{R}\Big) + \big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big)\mathcal{R} = \kappa M c^2 \delta(\vec{r})\delta^0_\mu\delta^0_\nu \qquad (5)$$

Here is what I got [Nguyen-2023c]

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2 - \frac{\kappa c^2}{48\pi\,\Lambda}\frac{M}{r}\right)c^2dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3}r^2 - \frac{\kappa c^2}{48\pi\,\Lambda}\frac{M}{r}} + r^2 d\Omega^2 \qquad (6)$$

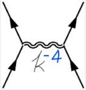
Surprise #2: **Newtonian limit exists in pure $\mathcal{R}^2$ gravity !** $\quad G = \dfrac{\kappa\, c^4}{96\pi\,\Lambda}$

A lot of studies (based on $f(R)$ or $\alpha\mathcal{R} + \beta\mathcal{R}^2 + \gamma$ Weyl) used locally flat background $\Rightarrow$ They failed to obtain Newtonian limit for pure $\mathcal{R}^2$ gravity $\Rightarrow$ Topics for a future talk

Alvarez-Gaume et al, "Aspects of quadratic gravity" [AlvarezGaume-2015]

On **de Sitter** background, spin-0 scalar excitation becomes **massless** too (instead of massive)

$\Rightarrow$ Long-range interaction instead of short-range Yukawa ...




$$\Rightarrow \quad U(\vec{r}) = (-i)^2 \int \frac{d^3k}{(2\pi)^2} \frac{e^{i\vec{k}.\vec{r}}}{k^4} = \frac{r}{8\pi}$$ Linear potential ...

$\Rightarrow$ New phenomenology ?

It is natural to include Weinberg-Salam into $\mathcal{R}^2$ gravity ...

$\Rightarrow$ Non-minimal coupling of $\mathcal{R}$ with Higgs [Salvio-2014]
or dilaton with Higgs [GarciaBellido-2011]

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$\mathcal{R}^2$ is a fourth-order theory ... $\quad \Box\mathcal{R} = 0 \;\Rightarrow\; \cancel{\mathcal{R} = \text{const}}$

Vacua with **non-constant** scalar curvature !

# Buchdahl 1962 program

# Buchdahl's derivation

If start with Schwarzschild gauge

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{7}$$

leading to two coupled ODE's for $A(r)$ and $B(r)$ ...

one at $4^{th}$ order, one at $3^{rd}$ order $\Rightarrow$ effectively $7^{th}$ order (!)

Buchdahl gauge: a new coordinate $x$ and

$$ds^2 = -A(x)dt^2 + B(x)dx^2 + \sqrt{\frac{A(x)}{B(x)}}d\Omega^2 \tag{8}$$

then the trace eqn $\Box\mathcal{R} = 0 \;\Rightarrow\; \frac{d^2\mathcal{R}}{dx^2} = 0$

$$\mathcal{R}(x) = \Lambda + k\,x \quad \Rightarrow \quad \text{Non-constant scalar curvature !} \tag{9}$$

# Buchdahl ODE

Cleverly make a series of transformations ... Buchdahl obtained an ODE for variable $q(r)$

$$r\frac{d^2q}{dr^2}+\left(\frac{2r^2}{1-r^2}-\frac{3k^2}{4q^2}\right)\frac{dq}{dr}=0 \tag{10}$$

**If** $q(r)$ can be found, then the $\mathcal{R}^2$ vacua would ensue !

Trouble: Buchdahl ODE is **non-linear** ...

He tried solving it via series expansion !

**5. – Solution of the equation.**

5'1. – Equation does not appear to be soluble in terms of known functions, nor does it appear to be reducible to a simpler form. It therefore seems appropriate to determine a solution in ascending powers of $t$, or in some similar form. The most convenient procedure seems to be to make the ansatz

$$q=c_0^{-1}\{1+\sum_{s=1}^{\infty}c_1^s q_s(t)\}\,, \tag{5.1}$$

My paper "Beyond SdS spacetimes #1" ...

# The general Buchdahl-inspired metric

I reworked the whole thing from field eqn (Benefits of hindsight!)

$$r\frac{d^2q}{dr^2}+\left(\frac{2\Lambda r^2}{1-\Lambda r^2}-\frac{3k^2}{4q^2}\right)\frac{dq}{dr}=0 \tag{11}$$

Introduce auxiliary variable $p(r)$ ... Evolution rules

$$\frac{dp}{dr}=\frac{3k^2}{4r}\frac{p}{q^2} \tag{12}$$

$$\frac{dq}{dr}=\left(1-\Lambda r^2\right)p \tag{13}$$

**The metric**

$$ds^2=e^{k\int\frac{dr}{r\,q}}\left\{p\left[-\frac{q}{r}dt^2+\frac{r}{q}dr^2\right]+r^2d\Omega^2\right\} \tag{14}$$

Ricci scalar

$$\mathcal{R}(r)=4\Lambda\, e^{-k\int\frac{dr}{r\,q}}\qquad \text{Non-constant !} \tag{15}$$

# Validation

Direct inspection, *ex post facto*

Check! See Maxima Online codes in [Nguyen-2023b]

And double check! See Mathematica notebook in [Shurtleff-2022]

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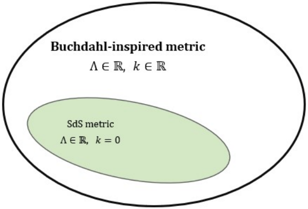
At $k = 0$ ...

$$\frac{dp}{dr} = 0 \qquad\qquad\qquad p = 1$$

$$\frac{dq}{dr} = \left(1 - \Lambda r^2\right) p \quad \Rightarrow \quad q = r - r_s - \frac{\Lambda}{3} r^3$$

$$\Rightarrow \quad \frac{q}{r} = 1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2$$

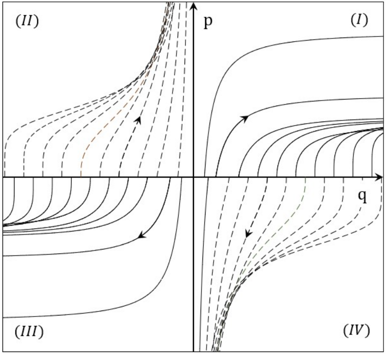
Recover Schwarzschild-de Sitter metric !



# Phase space ...

# Phase space

$$\frac{dp}{dr} = \frac{3k^2}{4r}\frac{p}{q^2}$$
$$\frac{dq}{dr} = \left(1 - \Lambda r^2\right) p$$




No sign flip in $g_{00}$ when $r \to 0^+$

$\Rightarrow$ Horizonless objects !

See my doc [Nguyen-2022c]

A linear potential ...

# A linear potential

Small $k$...

$$ds^2 = e^{k\int \frac{dr}{r\left(1-\frac{\Lambda}{3}r^2-\frac{r_s}{r}\right)}}\left\{-\left(1-\frac{\Lambda}{3}r^2-\frac{r_s}{r}\right)dt^2+\frac{dr^2}{1-\frac{\Lambda}{3}r^2-\frac{r_s}{r}}+r^2d\Omega^2\right\}+\dots \tag{16}$$

An effective potential

$$V_{\rm eff}(r) = -\frac{r_s+k}{2r} - \frac{k\Lambda}{2}r - \frac{k}{r}\dot{r}^2 \tag{17}$$

Surprise #3 ...

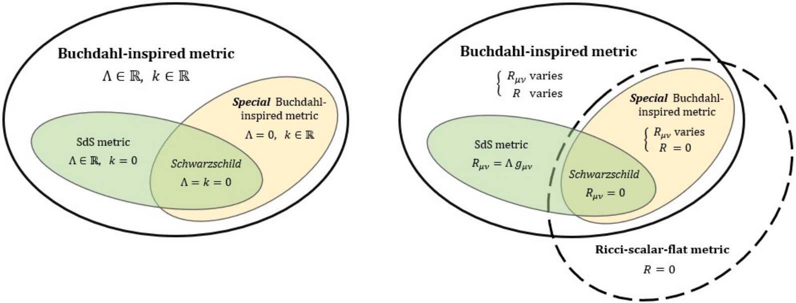
- Does the linear term relate to the massless spin-0 excitation ?
- Could the linear term account for flattening galactic rotation curves *bypassing* dark matter ?

$\Rightarrow$ Mannheim's phenomenological theory [Mannheim-2006] ...

See my doc [Nguyen-2022c]. Collaborations welcomed !

# Recap ...

# Recap




$\Lambda = 0$: *asymptotically flat* spacetimes

# Asymptotically flat Buchdahl-inspired metric

# My paper "Beyond SdS spacetimes #2": $\Lambda = 0$

Evolution rules

$$\begin{aligned} p_r &= \frac{3k^2}{4\,r}\,\frac{p}{q^2} \\ q_r &= p \end{aligned} \quad \Rightarrow \quad q_{rr} = \frac{3k^2}{4\,r\,q^2}\,q_r \tag{18}$$

Despite non-linear, solution (*serendipity!*)

$$r = |q - q_+|^{\frac{q_+}{q_+ - q_-}}\,|q - q_-|^{-\frac{q_-}{q_+ - q_-}} \tag{19}$$

$$q_\pm := \frac{1}{2}\left(-r_s \pm \sqrt{r_s^2 + 3k^2}\right) \tag{20}$$

$$p = \frac{(q - q_+)(q - q_-)}{r\,q} \tag{21}$$

Metric

$$\left|\frac{q - q_+}{q - q_-}\right|^{\frac{k}{q_+ - q_-}} \left\{ -\frac{p(q)\,q}{r(q)}dt^2 + \frac{r(q)}{p(q)\,q}dq^2 + r^2(q)d\Omega^2 \right\} \tag{22}$$

# The *special* Buchdahl-inspired metric

A new coordinate $\rho$

$$1-\frac{r_s}{\rho}=\text{sgn}\left(\frac{q-q_+}{q-q_-}\right)\left|\frac{q-q_+}{q-q_-}\right|^{\frac{r_s}{q_+-q_-}} \tag{23}$$

**Invertible!**

Metric

$$ds^2=\left|1-\frac{r_s}{\rho}\right|^{\tilde{k}}\left\{-\left(1-\frac{r_s}{\rho}\right)dt^2+\frac{r^4(\rho)\,dr^2}{\left(1-\frac{r_s}{\rho}\right)\rho^4}+r^2(\rho)d\Omega^2\right\} \tag{24}$$

with

$$r^2(\rho):=\frac{\zeta^2 r_s^2\left|1-\frac{r_s}{\rho}\right|^{\zeta-1}}{\left(1-\text{sgn}\left(1-\frac{r_s}{\rho}\right)\left|1-\frac{r_s}{\rho}\right|^{\zeta}\right)^2};\quad \zeta:=\sqrt{1+3\tilde{k}^2};\quad \tilde{k}:=\frac{k}{r_s} \tag{25}$$

# Superseding Schwarzschild...

At $k = 0$

$$\zeta = 1 \quad \text{and} \quad r(\rho) = \rho \qquad \forall \rho \in \mathbb{R} \tag{26}$$

$\Rightarrow$ Recover Schwarzschild metric !

Buchdahl parameter $k$ is new higher-derivative characteristic

For $k \neq 0$, the special metric is

- Ricci scalar flat: $\mathcal{R} = 0$
- Not Ricci flat: $\mathcal{R}_{\mu\nu} \neq 0$, hence non-Schwarzschild
- Asymptotically flat (approaching Minkowski as $r \to \infty$)

... describing $\mathcal{R}^2$ "structures" on asymptotically flat background

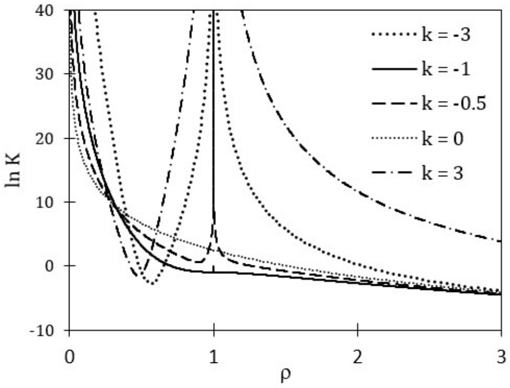
For the first time, an *analytical* tool to tackle $\mathcal{R}^2$ spacetimes !

Anomalous properties ...

# Kretschmann invariant

$$K := \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} \tag{27}$$

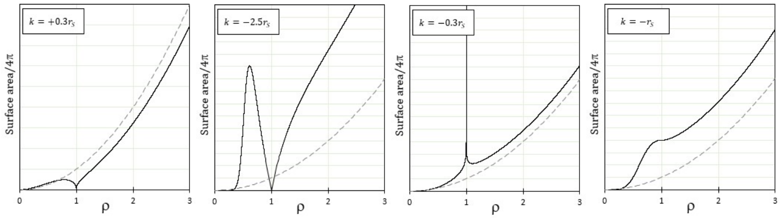
... too cumbersome to show here; see [Nguyen-2022b].




Curvature singularity at $\rho = r_s$ if $\tilde{k} \neq 0$ and $\tilde{k} \neq -1$ !

# Surface area of interior-exterior boundary

Surface area of "sphere" of radius $\rho$: $4\pi\left|1-\frac{r_s}{\rho}\right|^{\tilde{k}} r^2(\rho)$




Surface area of interior-exterior boundary:

- vanishes for $\tilde{k} \in (-\infty, -1) \cup (0, +\infty)$
- diverges for $\tilde{k} \in (-1, 0)$
- equals $16\pi r_s^2$ for $\tilde{k} = -1$
- equals $4\pi r_s^2$ for $\tilde{k} = 0$

Compatible with Kretschmann ...

Topology around interior-exterior boundary is altered !

$\Rightarrow$ Naked singularities or wormholes

# $\zeta-$Kruskal-Szekeres coordinates

## $\zeta-$tortoise coordinate

$$\frac{dr^*}{d\rho} = \frac{r^2(\rho)}{\rho^2\left(1 - \frac{r_s}{\rho}\right)} \tag{28}$$

## Solution

$$r^* = -\frac{\pi r_s}{\sin(\pi/\zeta)} + \frac{\zeta^2 r_s}{\zeta - 1}\left|1 - \frac{r_s}{\rho}\right|^{\zeta-1} {}_2F_1\left(2, 1 - \frac{1}{\zeta}; 2 - \frac{1}{\zeta}; \pm\left|1 - \frac{r_s}{\rho}\right|^{\zeta}\right) \tag{29}$$

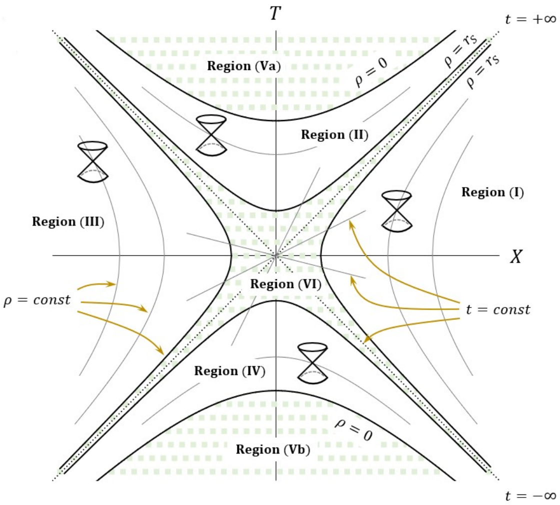
## $\zeta-$Kruskal-Szekeres (KS) coordinates

$$T^2 - X^2 = -\text{sgn}(\rho - r_s)e^{\frac{r^*(\rho)}{r_s}} \tag{30}$$

$$\frac{T}{X} = \left(\tanh\frac{t}{2r_s}\right)^{\text{sgn}(\rho - r_s)} \tag{31}$$

$$ds^2 = \left|1 - \frac{r_s}{\rho}\right|^{\tilde{k}}\left\{-4r_s^2 e^{-\frac{r^*(\rho)}{r_s}}\left|1 - \frac{r_s}{\rho}\right|\left(dT^2 - dX^2\right) + r^2(\rho)d\Omega^2\right\} \tag{32}$$

# $\zeta-$Kruskal-Szekeres diagram




"Gulf" as Region (VI) is a new feature

# Traversable Wormholes

# A few equivalent representations

For $\Lambda \in \mathbb{R}$, Azreg-Aïnou proposes a coordinate change $r(R)$

$$\frac{dr}{dR} = \frac{1}{p} \tag{33}$$

$$\frac{dp}{dR} = \frac{3k^2}{4}\frac{1}{r\,q^2} \tag{34}$$

$$\frac{dq}{dR} = 1 - \Lambda\, r^2 \tag{35}$$

The metric

$$ds^2 = e^{k\int\frac{dR}{\Psi\, r^2}}\left\{-\Psi\, dt^2 + \frac{dR^2}{\Psi} + r^2 d\Omega^2\right\} \tag{36}$$

$$\Psi := \frac{p\,q}{r} \tag{37}$$

# A few equivalent representations

For $\Lambda = 0$, in the exterior:

$$\left(\frac{q-q_+}{q-q_-}\right)^{\frac{\tilde{k}-1}{\zeta}}\left\{-\left(\frac{q-q_+}{q-q_-}\right)^{\frac{2}{\zeta}}dt^2+dq^2+(q-q_+)(q-q_-)d\Omega^2\right\} \quad (38)$$

For $\Lambda = 0$, in isotropic coordinate $\bar{r}$:

$$\left|\frac{\bar{r}-r_s/4}{\bar{r}+r_s/4}\right|^{\frac{2}{\zeta}(\zeta+\tilde{k}-1)}\left\{-\left|\frac{\bar{r}-r_s/4}{\bar{r}+r_s/4}\right|^{\frac{2}{\zeta}(2-\zeta)}dt^2+\zeta^2\left(1+\frac{r_s}{4\bar{r}}\right)^4\left(d\bar{r}^2+\bar{r}^2d\Omega^2\right)\right\} \quad (39)$$

symmetric w.r.t.

$$\frac{4\bar{r}}{r_s} \leftrightarrows \frac{r_s}{4\bar{r}} \quad (40)$$

Compared with Schwarzschild metric in Weyl's isotropic coordinate

$$-\left(\frac{\bar{r}-r_s/4}{\bar{r}+r_s/4}\right)^2dt^2+\left(1+\frac{r_s}{4\bar{r}}\right)^4\left(d\bar{r}^2+\bar{r}^2d\Omega^2\right) \quad (41)$$

# Traversable Morris-Thorne WH

MT ansatz [Morris/Thorne-1988]

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1-\frac{b(R)}{R}} + R^2 d\Omega^2 \tag{42}$$

Redshift function

$$e^{2\Phi(R)} = y^{\frac{2}{\zeta}(\eta+1)} \tag{43}$$

Shape function

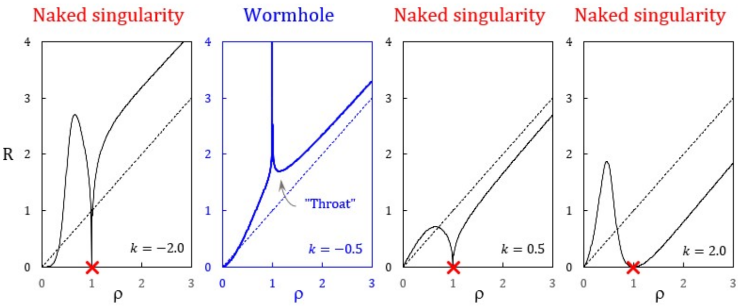
$$1-\frac{b(R)}{R} = \frac{1}{4y^2}\left((y^2+1)+\frac{\tilde{k}-1}{\zeta}(1-y^2)\right)^2 \geq 0 \tag{44}$$

with $y(R)$ implicit

$$R = (\zeta r_s)\frac{y^{\frac{\tilde{k}-1}{\zeta}+1}}{1-y^2}; \quad y := \left(1-\frac{r_s}{\rho}\right)^{\frac{\zeta}{2}} \tag{45}$$

# Traversable Morris-Thorne WH

$$R = (\zeta r_s)\frac{y^{\frac{\tilde{k}-1}{\zeta}+1}}{1-y^2}; \quad y := \left(1 - \frac{r_s}{\rho}\right)^{\frac{\zeta}{2}}$$




- For $\tilde{k} \in (-1, 0)$: $R(\rho)$ has a minimum in the exterior $\Rightarrow$ A traversable Morris-Thorne wormhole connecting 2 asymptotically flat sheets
- For $\tilde{k} \in (-\infty, -1) \cup (0, +\infty)$: a naked singularity at $\rho = r_s$
- For $\tilde{k} = -1$: a non-Schwarzschild structure with finite Kretschmann invariant at $\rho = r_s$

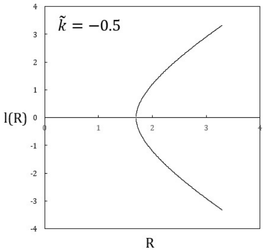
# Traversable Morris-Thorne WH

Four WH conditions:

1. $\Phi(R)$ be finite (no horizon)
2. $b(R_0) = R_0$ at the "throat" $R = R_0$
3. $b(R)/R \leq 1$ for $R \geq R_0$. Equal sign at "throat". $g_{RR}$ does not flip sign as $R \to R_0^+$.
4. Asymptotically flat: $b(R)/R \to 0$ as $R \to +\infty$

The proper radial distance $l(R)$

$$l(R) = \pm \int_{R_0}^{R} \frac{dR}{\sqrt{1 - \frac{b(R)}{R}}} = \pm \frac{\zeta r_s}{1 + \frac{\tilde{k}-1}{2\zeta}} \left[ y^{2+\frac{\tilde{k}-1}{\zeta}} \, {}_2F_1\left(2, 1 + \frac{\tilde{k}-1}{2\zeta}; 2 + \frac{\tilde{k}-1}{2\zeta}; y^2\right) - \text{const} \right] \quad (46)$$



# Weak Energy Condition

$$G_{00} = \tilde{k}(\tilde{k}+1)\,\frac{\left(1-\operatorname{sgn}\left(1-\frac{r_s}{\rho}\right)\left|1-\frac{r_s}{\rho}\right|^{\zeta}\right)^4}{2\zeta^4 r_s^2\left|1-\frac{r_s}{\rho}\right|^{2\zeta-2}} \tag{47}$$

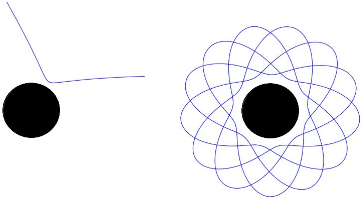
$$\text{Violation of WEC: } G_{00} < 0 \quad \forall r \quad \Leftrightarrow \quad \tilde{k} \in (-1, 0)$$

$$\Leftrightarrow \quad \text{there is a wormhole}$$

Special Buchdahl-inspired wormhole and Campanelli-Lousto wormhole in Brans-Dicke gravity share similarities [Campanelli-1993, Nguyen-2023a]

The scalaron degree of freedom acts as an "exotic" form of matter to violate WEC

# Orbits, shadows, etc.



Perihelion and aphelion are not on opposite sides of massive object !

Model Sagittarius A* as $\mathcal{R}^2$ wormhole ? Extracting bound for $\tilde{k}$ ?

Note: $\tilde{k}$ is not a parameter of the theory but a parameter of the solution. It can vary from one system to another, unlike Brans-Dicke parameter ...

Azreg-Aïnou extended my work to axisymmetric setup and applied for Sgr A* shadow [AzregAinou-2023]. Per his courtesies:

$$-0.129 \leq \tilde{k} \leq -0.033$$

$\Rightarrow$ Is Sgr A* a wormhole ? Buckle up for interstellar travel !

Call Elon ...

# Problems with a no-go "theorem"

*Gaps in the generalized Lichnerowicz theorem [Nelson-2010]:*

Trace eqn:

$$\Box\mathcal{R} = 0 \tag{48}$$

$$\Rightarrow \int d^3x(\nabla\mathcal{R})^2 = \int d^3x\,\nabla\left(\mathcal{R}\nabla\mathcal{R}\right) = \oint_{S_\infty} d\vec{S}\left(\mathcal{R}\nabla\mathcal{R}\right) - \oint_{S_h} d\vec{S}\left(\mathcal{R}\nabla\mathcal{R}\right) \tag{49}$$

Assuming:

(i) $\nabla\mathcal{R}$ falls to zero rapidly enough when $r \to \infty$
(ii) $\nabla\mathcal{R}$ diverges not too rapidly when $r \to$ horizon

$$\Rightarrow \quad (\nabla\mathcal{R})^2 = 0 \quad \Rightarrow \quad \mathcal{R} = \text{const} \tag{50}$$

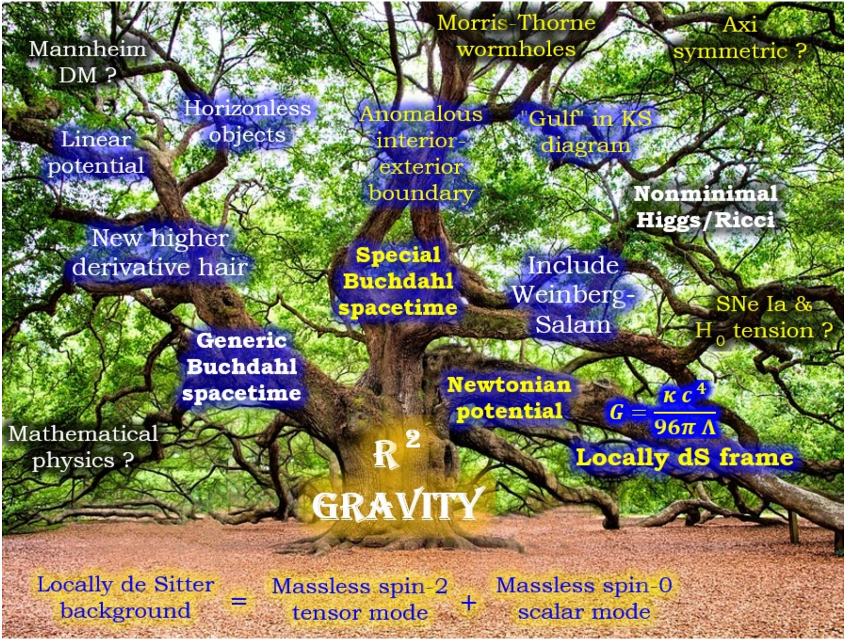
Buchdahl-inspired solution with non-constant scalar curvature disproves this no-go "theorem"

Assumptions (i) and (ii) are overly strong !

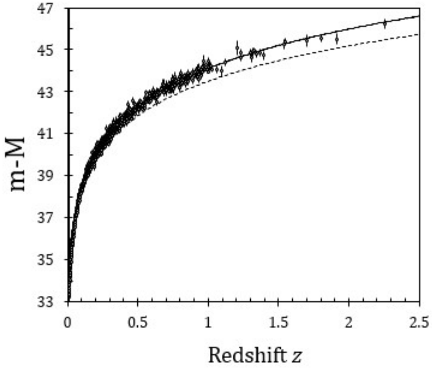
$\Rightarrow$ Modifications to Lü-Perkins-Pope-Stelle solution in Einstein-Weyl gravity (See Lü *et al* in [Nelson-2010] and my doc [Nguyen-2022c])

# $\mathcal{R}^2$ gravity:

# New horizons ahead


Mannheim DM ?
Morris-Thorne wormholes
Axi symmetric ?
Horizonless objects
Linear potential
Anomalous interior-exterior boundary
"Gulf" in KS diagram
Nonminimal Higgs/Ricci
New higher derivative hair
Special Buchdahl spacetime
Include Weinberg-Salam
SNe Ia & $H_0$ tension ?
Generic Buchdahl spacetime
Newtonian potential
$G = \frac{\kappa c^4}{96\pi\Lambda}$
Locally dS frame
Mathematical physics ?
$R^2$
GRAVITY
Locally de Sitter background = Massless spin-2 tensor mode + Massless spin-0 scalar mode

# Accounting for SNe Ia (Pantheon)




Let Ricci scalar couple with Higgs sector ... $\mathcal{R}\,\Phi^\dagger\Phi$

**Plus extra care** to handle the locally de Sitter background ...

Topics for a future talk. Writeup in preparation. Collaborations welcomed !

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# $\mathcal{R}^2$ action in Einstein frame

In Jordan frame

$$\int d^4x\sqrt{-g}\,\frac{1}{2\kappa}\mathcal{R}^2 \tag{51}$$

Auxiliary scalar field $\Omega$

$$\int d^4\sqrt{-g}\left[\Omega\mathcal{R}-\frac{\kappa}{2}\Omega^2\right] \tag{52}$$

Conformal transform

$$g_{\mu\nu}=\Omega^{-1}\,\tilde{g}_{\mu\nu} \tag{53}$$

then

$$\sqrt{-g}\Omega\mathcal{R}=\left(\cancel{\Omega^{-2}}\sqrt{-\tilde{g}}\right)\cancel{\Omega\Omega}\left[\tilde{\mathcal{R}}+3\tilde{\Box}\ln\Omega-\frac{3}{2}\left(\tilde{\nabla}\ln\Omega\right)^2\right] \tag{54}$$

and in "Einstein frame"

$$\int d^4x\sqrt{-\tilde{g}}\left[\tilde{\mathcal{R}}\;+\;3\cancel{\tilde{\Box}\ln\Omega}\;-\frac{3}{2}\left(\tilde{\nabla}\ln\Omega\right)^2-\frac{\kappa}{2}\right] \tag{55}$$

*surface term*