Cosmological Parametrization: A Model-Independent approach in Cosmological Modelling and Reconstructing Cosmic Evolution

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OUTLINE OF TALK

- **Introduction:**
- **Einstein Field Equations:**
- **Exact Solution:**
- **Dark Energy:**
- **Cosmological Parameters:**
- **Cosmological Parametrization:**
- **Exemplification:**
  - **Modelling with Dark Energy**
  - **Modelling with Modified gravity:**
- **Conclusion:**

INTRODUCTION

- **Cosmology** → **Study of the origin, evolution, structure formation, dynamics and ultimate fate of the Universe.**
- **High precision observations and new scientific discoveries at theoretical ground make cosmology an exciting field of research.**
- **Physical Universe is well described by** → **Gravity**
- **Gravity** → **Play a major role in the creation and structure formation and also regulates the dynamics of the Universe.**
- **Best description of Gravity** → **General Relativity**

- 100+ years of journey of GR.
- Encountered with many hurdles, axed many times with observational discoveries (1929 by Hubble's observation, 1965 by CMB observation, 1998 by SN Ia observation ...). But, stood high with the successes.

Successes of general relativity through experiment and observation.

- **SM → suffers from initial singularity problem, age problem, cosmological constant problem, flatness problem, hierarchy problem etc. (Fundamental problems associated to SM)**
- **Moreover, late-time cosmic acceleration can not be explained within the framework of GR.**
- **Modifications of general relativity (modified gravity models) required to address few of those problems → Massive gravity, Gauss–Bonnet gravity, $f(R)$, $f(T)$, $f(R,T)$ gravities are names of a few among the various alternative theories proposed in the past few years.**

VARIOUS THEORIES OF GRAVITY

- Affine gauge theory
- Alternatives to general relativity
- AQUAL
- Bi-scalar tensor vector gravity
- Bimetric gravity
- Brans–Dicke theory
- Chasles' theorem (gravitation)
- Chronology protection conjecture
- Composite gravity
- Conformal gravity
- Cosmological constant
- Dark fluid
- Democratic principle
- DGP model
- Einstein aether theory
- Einstein–Cartan theory
- Emergent gravity
- Entropic gravity
- An Exceptionally Simple Theory of Everything
- Extended theories of gravity
- F(R) gravity
- Fermat’s and energy variation principles in field theory
- Gauge gravitation theory
- Gauge theory gravity
- Gauge vector–tensor gravity
- Gauss–Bonnet gravity
- Gauss's law for gravity
- General relativity

- Geometrodynamics
- Graviscalar
- Gravitational field
- Hayward metric
- Higher-dimensional Einstein gravity
- Higher-dimensional supergravity
- History of gravitational theory
- Hořava–Lifshitz gravity
- Hoyle–Narlikar theory of gravity
- Induced gravity
- Kaluza–Klein theory
- Large extra dimension
- Le Sage's theory of gravitation
- Loop quantum gravity
- Lovelock theory of gravity
- Mach's principle
- Massive gravity
- Mechanical explanations of gravitation
- Metric-affine gravitation theory
- Modified Newtonian dynamics
- Newton–Cartan theory
- Newton's law of universal gravitation
- Nonsymmetric gravitational theory
- Nordström's theory of gravitation
- Nuts and bolts (general relativity)
- Parameterized post-Newtonian formalism
- Plebanski action
- Polarizable vacuum
- Pressuron
- Quantized inertia
- Quantum gravity
- Rainbow gravity theory
- Scalar theories of gravitation
- Scalar–tensor theory
- Scalar–tensor–vector gravity

- Semiclassical gravity
- Social gravity
- Stochastic electrodynamics
- Supergravity
- Teleparallelism
- Tensor–vector–scalar gravity
- Theory of everything
- Twisted geometries
- Twistor theory
- Unified field theory
- Whitehead's theory of gravitation
- World crystal
- Yilmaz theory of gravitation

Some more theories of the Universe also proposed other than gravity in the past few decades.

Examples:

- Steady State theory
- Quasi steady state theory
- String theory
- Biocentrism
- Multiverse theory
- The Bouncing model
- Cyclic Universe theory
- The Black hole Universe theory
- Biocosmology (Multiverse, Life and Consciousness)

f(R), f(R,T), f(T), f(T,G), f(G), f(Q), f(Q,T), f(R,L_m) theories of gravity

We start our discussion with the Einstein Field Equations in GR.

EINSTEIN FIELD EQUATIONS

Einstein Field Equations are given by

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where, $G_{\mu\nu}$ is the Einstein Tensor, $T_{\mu\nu}$ is the stress energy tensor, G is the Newton's gravitational constant.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar. $g_{\mu\nu}$ is the metric tensor.

LHS of EFEs describes geometry of the Universe and RHS the matter in the Universe.

LHS of EFE → MATHEMATICS
RHS of EFE → PHYSICS

$$R_{\mu\nu} - (1/2)Rg_{\mu\nu} = (8\pi G)T_{\mu\nu}$$

Volume of Space-Time Curvature

The density and flux of energy and momentum in the spacetime manifold

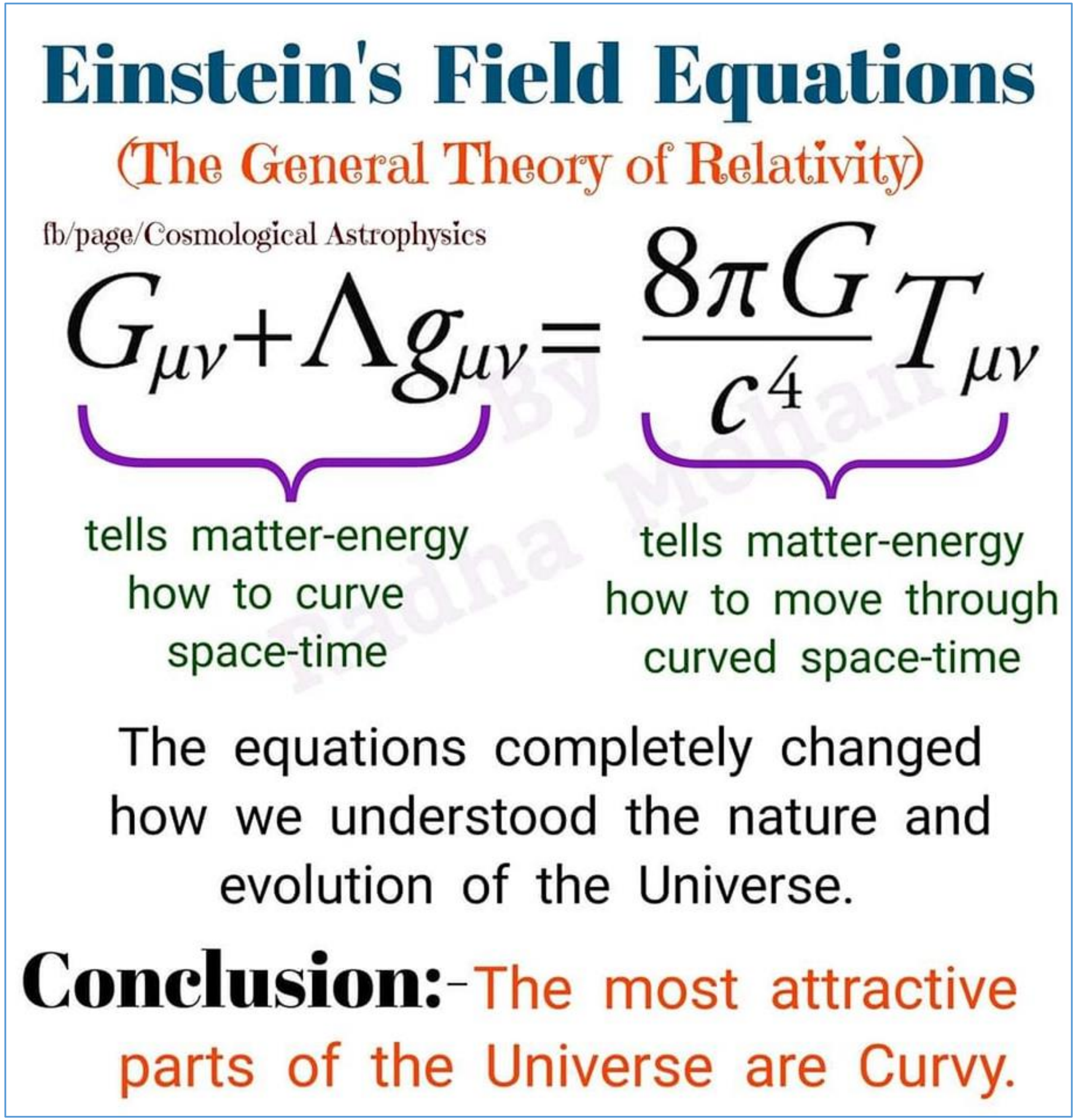
Forces experienced as an object moves within the Space-Time Curvature

The force field's function on the Space-Time manifold

SOME FEATURES OF EFES

- **Einstein Field Equation** is a **tensor equation** relating a set of symmetric 4×4 tensors. Each tensor has 10 independent components. So, there are 10 numbers of 2nd order nonlinear partial differential equations in 4 independent variables. Bianchi identities reduce the number of independent equations from 10 to 6.
- Trace ⇨ $g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R \Longrightarrow G = R - \frac{1}{2}nR = \frac{2-n}{2}R$. For $n = 4, G = -R$. So, Einstein tensor is also called the *trace-reversed Ricci tensor.*

- EFEs are fundamental equations of GR providing the equations of motion for the spacetime metric in the presence of matter.
- Despite the simple appearance ($G_{\mu\nu} = 8\pi G T_{\mu\nu}$) of the equations they are in fact quite complicated.



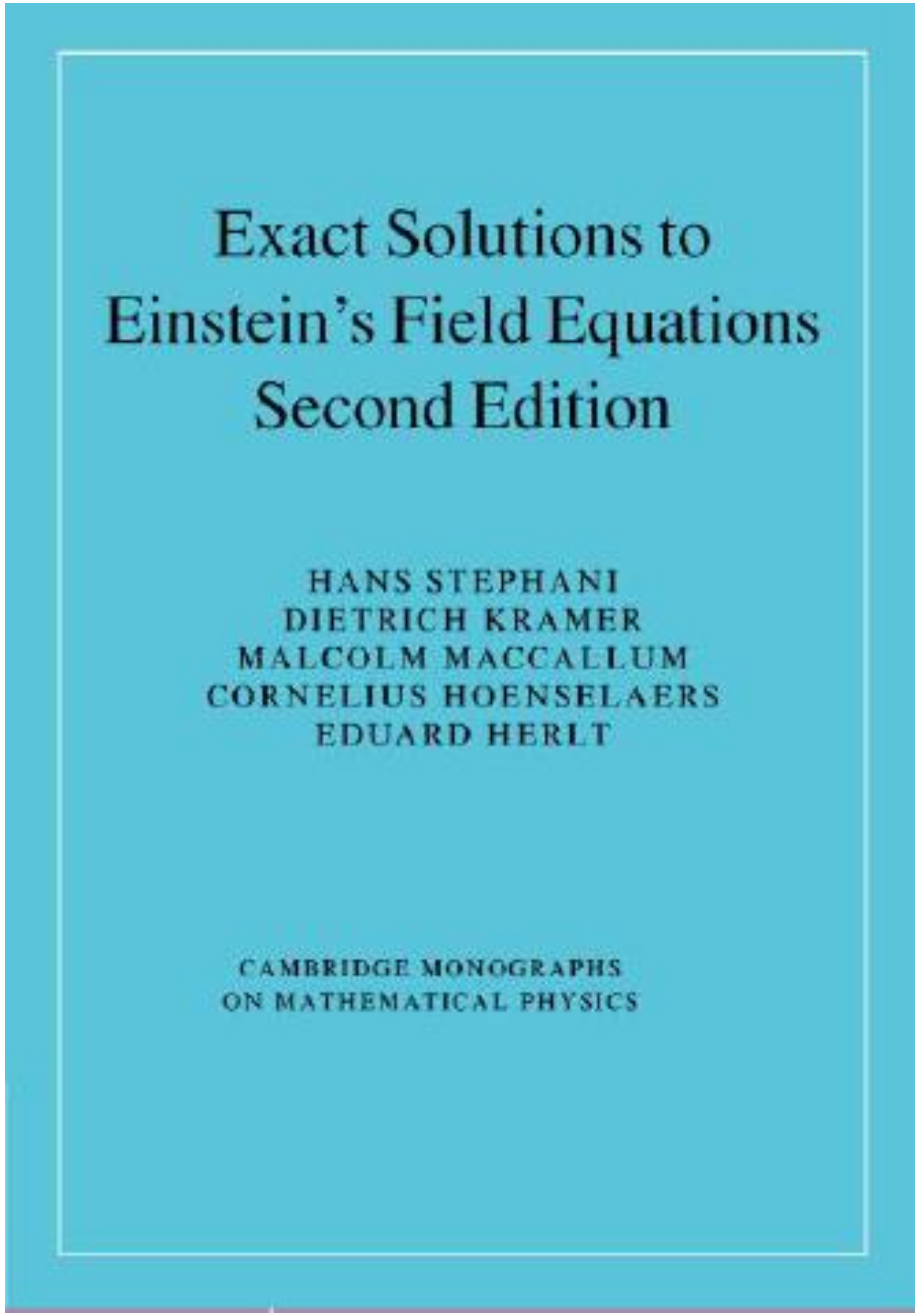
- **The EFEs reduce to Newton's law of gravity by using both weak-field approximation and the slow-motion approximation. In fact, the constant G appearing in EFEs is determined by making these two approximations.**
- **If the energy momentum tensor is that of an electromagnetic field in free space, then the EFEs are called the Einstein-Maxwell equations (with cosmological constant).**
- **If the energy-momentum tensor is zero, then the FEs are refers to as the vacuum field equations.** $R_{\mu\nu} = 0$.
- **Next we discuss the solution of EFEs.**

EXACT SOLUTION

- **The solution of the EFEs are metrics of spacetime. The solutions are hence called metrics. These metrics describe the structure of the spacetime including the inertial motion of objects in the spacetime.**
- **As the FEs are non-linear, they cannot always be completely solved (without making approximations).**
- **However, approximations are usually made in these cases (commonly referred as post-Newtonian approximations.).**
- **Even so, there are numerous cases where the field equations have been solved completely and those are called exact solutions.**

- The study of exact solutions of EFEs is one of the activities of cosmology.
- It leads to the prediction of black holes and to different models of evolution of the Universe.
- Exact solutions of Einstein field equations is important in studying the nature & behavior of the physical Universe.
- The first exact solution of the EFEs is the Schwarzschild exterior solution, wherein the prefect fluid equation of state was considered as a supplementary condition.

- **Despite of the high non linearity of the EFEs, various exact solutions are obtained for static and spherically symmetric metrics.**
- **Einstein's static solution, de-Sitter solution, Tolman's solutions, Adler's solutions, Buchdahl's solution, Vaidya and Tikekar solution, Durgapal's solutions, Knutsen's solutions and many more well-known solutions of EFEs are obtained which are summarized in the literature (D. Kramer et al., "Exact solutions of Einstein's equations", Cambridge, (1980).).**
- **All those phenomenological cosmological models explain the Universe theoretically very well.**



Applications of exact solutions in Astrophysics

- Slowly rotating stars and planets: The Schwarzschild solution
- Static black holes: The Schwarzschild solution
- Star interiors: e.g. Tolman, Buchdahl, Heintzmann solutions
- Neutron stars: Tolman VII and Durgapal solutions
- Rotating black holes: The Kerr-(Newman) solution
- Gravitational waves: gravitational plane wave exact solution
- Standard model of cosmology: The Friedmann-Lemaitre-Robertson-Walker solution
- Inhomogeneous Cosmological models: e.g. Lemaitre-Tolman-Bondi solutions, Szekeres solutions, Oleson solutions

SOLUTIONS OF EFEs ARE USUALLY OBTAINED

- **By assuming symmetries on the metric and other simplifying restrictions.**
- **Two basic assumptions are that galaxies are homogeneously distributed on galaxies larger than 50 Mpc and that the Universe is isotropic around us on angular scales larger than about 10 degrees.**
- **With this simplified assumptions that our place is the Universe is not special at all, then isotropy around all its points is inferred.**
- **Finally, there is a theorem in geometry, which tells us that if every observer sees the same picture of the Universe when looking at different directions, then the Universe is homogeneous.**

- These assumptions boil down into the (Friedmann-) Robertson-Walker metric. Our Universe can be viewed as an expanding, isotropic and homogeneous spacetime, and the line element reads;

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right]$$

$k = \pm 1$ is the curvature parameter describing the geometry of spatial sections and $a(t)$ is the scale factor. We have chosen the units with $c = 1$. The coordinates (r, θ, ϕ) are commoving coordinates.

- Up to now, we have discussed about the geometry (LHS of EFEs) part. Following we, discuss the matter (RHS of EFEs).

- Given a source energy-momentum tensor, an exact solution to the Einstein equations, where the spacetime metric functions are expressed in terms of elementary or well-known special functions.
- We have seen that simplifications in geometry are required to model the Universe. In the same spirit reduction of sophistication in the description of matter/energy is also required.
- Simplicity on the one hand, and consistency with observations in the other, suggest adopting the perfect fluid picture.

- **So, for a perfect fluid source i.e. if we assume the matter content in the Universe is filled with perfect fluid, then the energy momentum tensor takes the form:**

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

where ρ, p, u_μ representing the energy density, pressure and velocity of the fluid.

- **Then the Einstein Field Equations in the FLRW background reads**

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{a^2} \qquad (1)$$

$$\dot{H} = -4\pi G \sum_i (\rho_i + p_i) + \frac{k}{a^2} \qquad (2)$$

where H is the Hubble parameter.

- These are two independent equations known as Friedmann equation and Raychaudhuri equation.

- From Einstein equations one can derive other two important equations, the energy conservation equation and acceleration equation, which tells us about the evolution of the spatial separation between geodesics.

$$\dot{\rho} + 3H\sum_i(\rho_i + p_i) = 0 \qquad (3)$$

And

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i) \qquad (4)$$

- **These preliminaries suggest the interplay between the matter/energy content of the Universe and its geometry have a crucial influence in its final fate.**
- **Out of the four equations above, only two are independent with three variables a, ρ, p.**

- **In order to find a consistent solution, we need to close the system and we need one more equation.**
- **Since, we have considered the perfect fluid source, a relation between it's pressure and density is natural and the simplest form of the equation of state would be a linear one i.e.**

$$\boldsymbol{p = w\rho} \quad (5)$$

where, different values of the $\boldsymbol{w}$ describe different matter content in the Universe.

- **Now, we have three equations with three variables and we can have a consistent solution to the EFEs.**

- For flat geometry ($k = 0$), EFEs can be solved for;

  Electromagnetic Radiation (Photon): $w = 1/3$

  solution will be: $a(t) \propto t^{\frac{1}{2}}$ $\rho \propto a^{-4}$

  Incoherent matter (Cosmic Dust): $w = 0$

  solution will be: $a(t) \propto t^{\frac{2}{3}}$ $\rho \propto a^{-3}$

- In both the cases, we get the deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2} = +1$ (**radiation**) and $q = +\frac{1}{2}$ (**dust**) > 0 implying the decelerated expansion of the Universe.

- Similarly, for a constant energy density ($\rho = Constant \Longrightarrow \dot{\rho} = 0$), we have from the continuity equation, $p = -\rho$ or $w = -1$ and Hubble parameter comes out to be constant. The scale factor follows $a \propto e^{Ht}$ and $q = -1$. It's the de-Sitter Universe.
- Moreover, for a static Universe ($a = constant$) and we have $H = 0$, $\ddot{a} = 0$. From equations (2) and (5), we have $\rho = -3p = \frac{3k}{8\pi G a^2} \Rightarrow k = 1$.
- However, the linear equation of state $p = w\rho$ is not the only choice. EoS may be quadratic or other forms too. In general $p = f(\rho)$.

- If we observe the technique of finding a solution of the EFEs, we can see:
- There are 3 variables $a(t)$, $\rho(t)$, $p(t)$ that may take any functional form. For example:

$a(t) = Constant, a(t) = e^{\alpha t}, a(t) = t^n,$ or any complicated function.
$\rho(t) = Constant, \rho(t) = e^{\alpha t}, \rho(t) = t^n,$ or any complicated function.
$p(t) = Constant, p(t) = e^{\alpha t}, p(t) = t^n,$ or any complicated function.
Or a relation between the variables e.g. $p = f(\rho)$

- So, mathematically, we have choices to consider a functional form of the variables (time t or redshift z) or a relation between them. They may contain either one or more arbitrary parameters α, β, n, w etc.

Various functional forms of $f(\rho)$ considered:

Pressure $p(\rho)$, $p(z)$

$p(\rho) = w\rho$ (Perfect fluid EoS)

$p(\rho) = w\rho - f(H)$ (Viscous fluid EoS)

$p(\rho) = w\rho + k\rho^{1+\frac{1}{n}}$ (Polytropic gas EoS)

$p(\rho) = \frac{8w\rho}{3-\rho} - 3\rho^2$ (Vanderwaal gas EoS)

$p(\rho) = -(w+1)\frac{\rho^2}{\rho_P} + w\rho + (w+1)\rho_\Lambda$ (EoS in quadratic form)

$p(\rho) = -\frac{B}{\rho}$ (Chaplygin gas EoS)

$p(\rho) = -\frac{B}{\rho^\alpha}$ (Generalized Chaplygin gas EoS)

$p(\rho) = A\rho - \frac{B}{\rho^\alpha}$ (Modified Chaplygin gas EoS)

$p(\rho) = A\rho - \frac{B(a)}{\rho^\alpha}$ (Variable modified Chaplygin gas EoS)

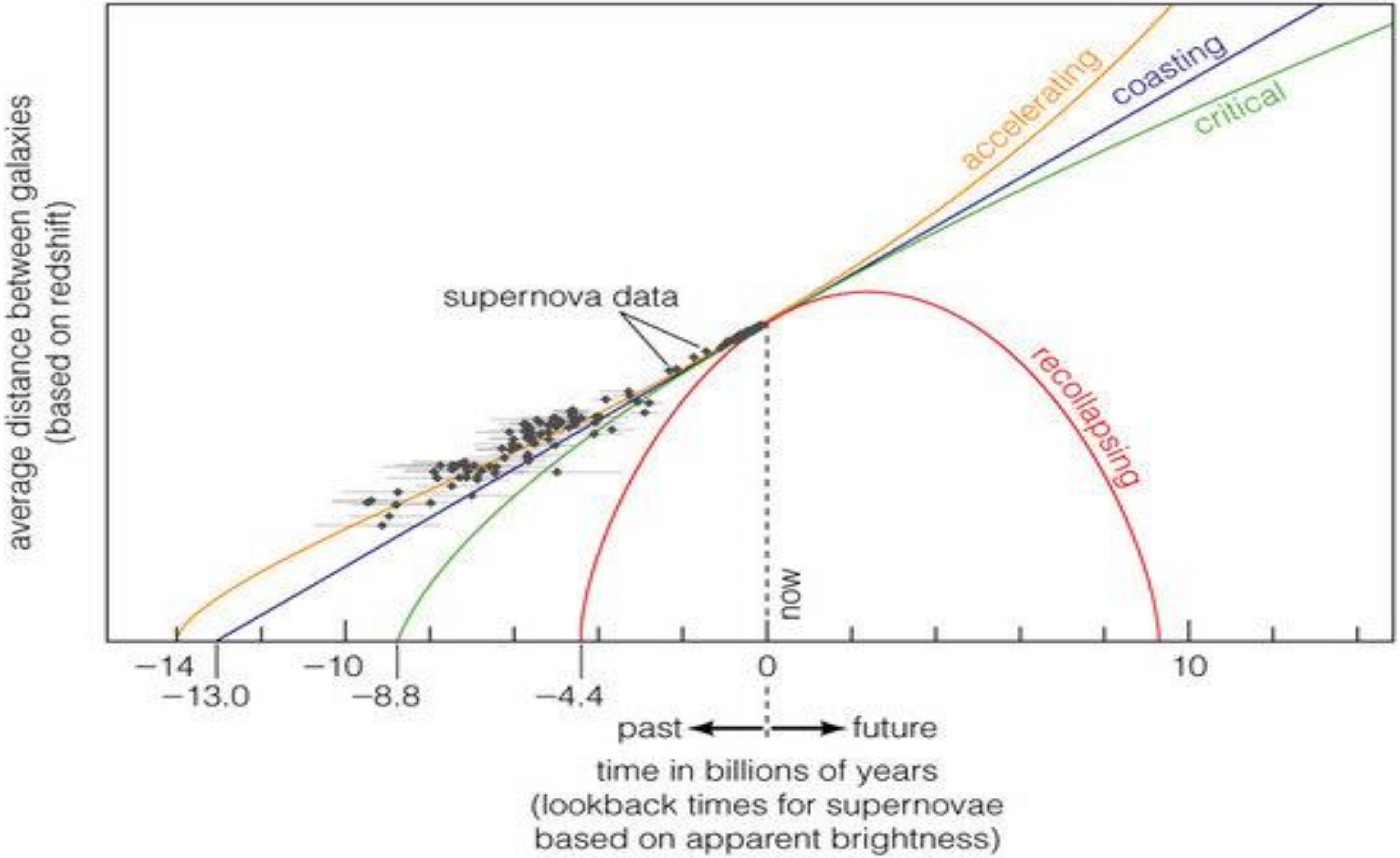
$p(\rho) = A(a)\rho - \frac{B(a)}{\rho^\alpha}$ (New variable modified Chaplygin gas EoS)

$p(\rho) = -\rho - \rho^\alpha$ (DE EoS)

- **With the linear form of EoS $p = w\rho$, different values of the w describe different matter content in the Universe other than radiation/ dust matter.**
- **For example: For $w = -1$, it represents vacuum energy, for $w < -\frac{1}{3}$ $(\neq -1)$, represents quintessence and $w < -1$, represents phantom.**
- **In the next section, we discuss these in relation to the recent discovery of Late-time cosmic acceleration and our motivation of (of this talk) COSMOLOGICAL PARAMETRIZATION to solve EFEs.**

DARK ENERGY

- **Before 1990's, it was a common understanding that the expansion of the Universe is slowing down due to attractive gravity and theorists were working on models of the Universe with decelerating expansion.**
- **Moreover, solutions of standard model with normal matter sources are found with a positive value of deceleration parameter.**
- **But, the observations on Type Ia supernovae suggested accelerating expansion of the Universe.**


average distance between galaxies
(based on redshift)
accelerating
coasting
critical
supernova data
recollapsing
now
−14
−10
0
10
−13.0
−8.8
−4.4
past
future
time in billions of years
(lookback times for supernovae
based on apparent brightness)

- **This is the birth of a new phase of cosmological studies and cosmologists started to think about late-time cosmic acceleration with an earlier phase of deceleration.**
- **This discovery of cosmic acceleration again shook the foundation of general relativity.**
- **Later on the idea of cosmic acceleration received more and more evidence by many other observations as well as at theoretical ground.**

- **DE was supported by some independent observations e.g. the BOOMERanG, Maxima, CMBR, BAO, 2dF Galaxy Redshift Survey, DES etc.**
- **Much more precise measurements from WMAP have continued to support the SM and give more accurate measurements of some cosmological parameters.**
- **The idea of late-time cosmic acceleration is now playing a major role in precision cosmology.**

- Now, the question arises, how to get accelerating expanding solutions?
- Within the background of GR, it is difficult to get acceleration with normal matter source ($\rho > 0, p > 0$).
- Two simplest ways are ⇒ Modifying the left hand side of EFEs (Geometric modification) and inserting additional term in the right hand side of EFEs with high negative pressure (Physical modification).
- These two modifications produced a plethora of cosmological models in the past twenty five years leading to accelerating expanding solutions.

- **However, these are not the only possibilities and is an open question till. One can deduce accelerating solutions without any modification or incorporating extra degrees of freedom too.**
- **The simplest and most significant way to get accelerating solutions is by incorporating the Einstein's cosmological constant in the RHS of EFEs but as a negative matter source.**
- **Einstein introduced cosmological constant (CC) into field equations as he was convinced with static Universe. The modifications of EFEs can be seen as:**

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein's original equation

Law of an expanding universe

All matter and energy in the universe

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein's modified FE

Law of an expanding universe

Cosmological constant

All matter and energy in the universe

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} - \bar{\rho}_{DE}\, g_{\mu\nu})$$

Law of an expanding universe

All matter and energy in the universe

- The modified EFEs with CC can be written as

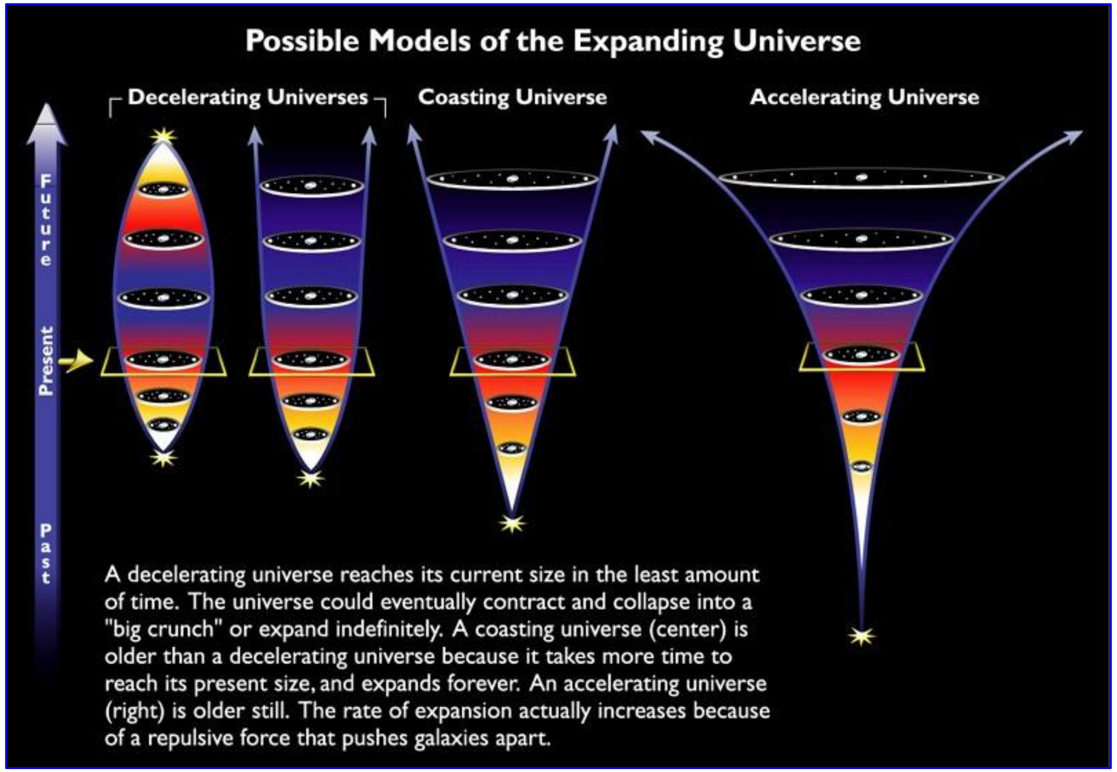
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad (6)$$

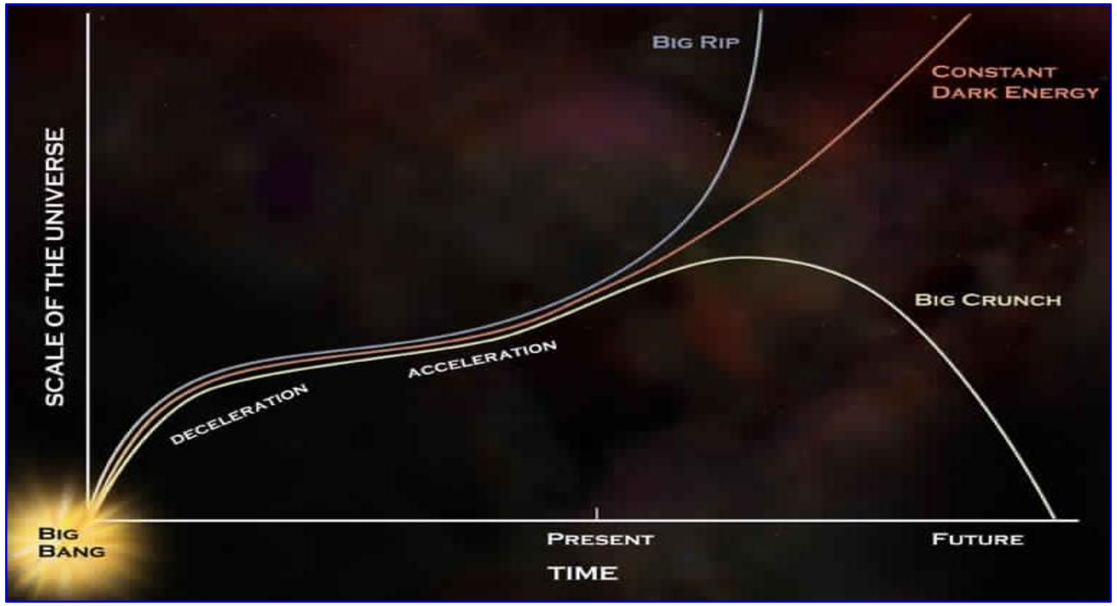
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \qquad (7)$$

- For a dust ($p = 0$) dominated Universe in presence of CC, Einstein's static Universe corresponds to, $\rho = \frac{\Lambda}{4\pi G}, \frac{k}{a^2} = \Lambda$. Λ must be positive since $\rho > 0$ implying for a static universe $k = +1$ (closed) with a radius $a = \frac{1}{\sqrt{\Lambda}}$.

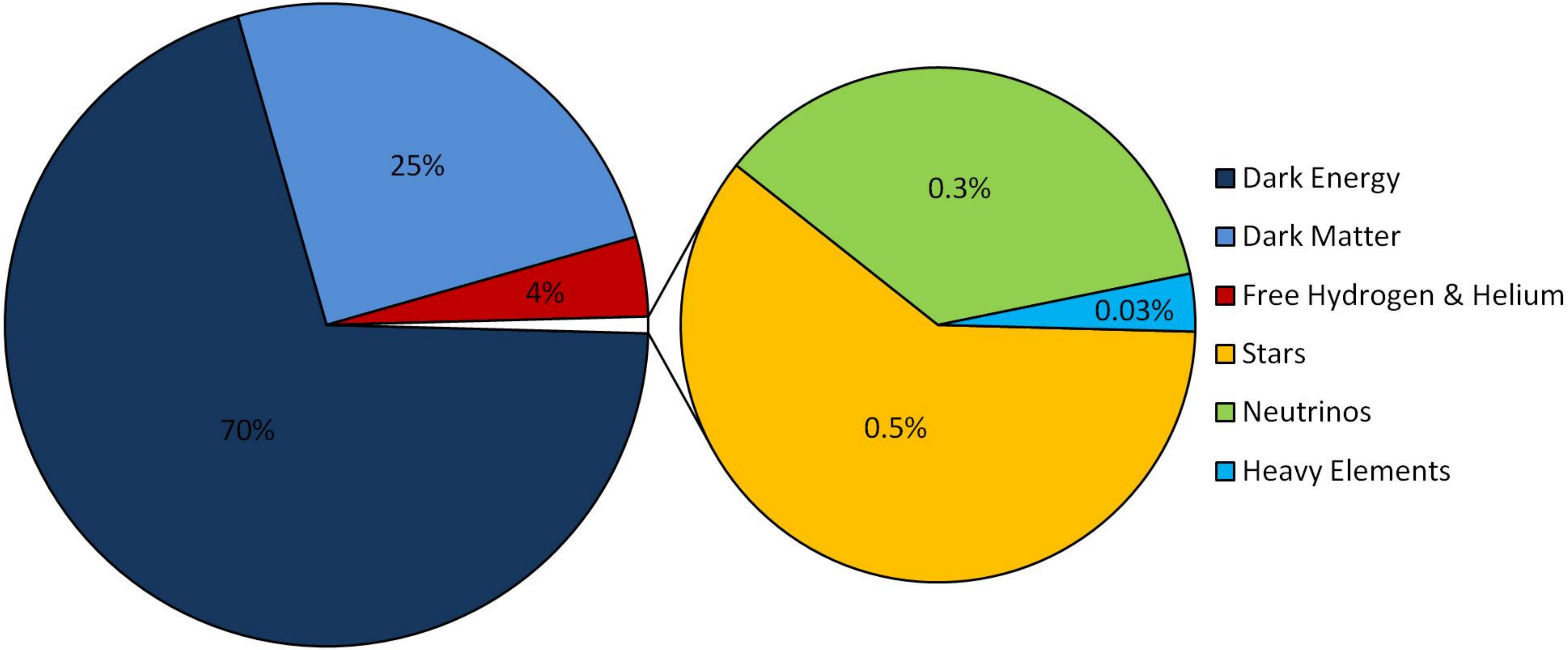
- **Although, ΛCDM model is consistent with observations, it suffers from the long standing Cosmological Constant problem due to the non-dynamical equation of state of Λ.**
- **The incorporation of DE resolved the problem of late-time cosmic acceleration largely but a new problem arose, "what is the suitable candidate of dark energy?", due to the non evolving nature of the cosmological constant, which is plagued with fine tuning problem.**
- **So, models with dynamical EoS are explored.**

- The **data from the globular cluster** also reveals that the age of certain objects in the Universe could be larger than the present estimated age of the Universe in standard model with normal matter.
- As of now, the only known resolution of this puzzle is provided by invoking the concept of late-time cosmic acceleration.





- **Although, the nature of DE is unknown but it is generally considered to be homogeneous and permeates all over space.**
- **The total energy budget in the Universe is estimated as:**



Dark Energy Modelling Beyond ΛCDM

- **There were several attempts to solve the long standing cosmological constant problem even before the discovery of cosmic acceleration by varying Λ.**
- **In this direction, authors have considered some variation laws for the cosmological constant in the past forty years, commonly known as "Λ-varying cosmologies" or "Decaying vacuum cosmologies". Following is list of such decay laws of Λ.**

$$\Lambda \sim a^{-n}$$
$$\Lambda \sim H^{n}$$
$$\Lambda \sim t^{n}$$
$$\Lambda \sim q^{n}$$
$$\Lambda \sim \rho$$
$$\Lambda \sim e^{-\beta a}$$
$$\Lambda \sim C + e^{\beta t}$$

$\Lambda = \Lambda(\mathrm{T})$, T is Temperature
$$\Lambda = 3\beta H^2$$
$$\Lambda = 3\beta H^2 + \alpha a^{-2}$$
$$\Lambda = \beta \frac{\ddot{a}}{a}$$
$$\Lambda = 3\beta H^2 + \alpha \frac{\ddot{a}}{a}$$
$$\frac{d\Lambda}{dt} = \beta\Lambda - \Lambda^2$$

Here, α, β, n, C are constants.

- **The problem can also be alleviated with a dynamically evolving scalar field.**
- **Dynamically decaying Λ, vacuum energy have also been used to explain late-time cosmic acceleration.**
- **A variety of scalar field models have been proposed to describe the late-time cosmic acceleration including quintessence, phantoms, K-essence, Tachyon scalar fields and some more.**
- **Some other DE models with scalar field are Chameleons, Galileons, Holographic scalar field and non-minimally coupled scalar field.**

- A quite different approach for the description of cosmic acceleration is to consider the Chaplygin gas EoS (and its modifications), Polytropic gas EoS, Vander wall's fluid and bulk viscous fluid.
- However, the search for suitable candidate of dark energy is still an open question.
- Here, we shall discuss the theoretical approach to some dark energy models in classical general relativity and also discuss the reconstructions of these models with cosmological parametrization.

Scalar Field Model of dark Energy

There are observations which constrain the value of EoS parameter w today refers towards the time evolution of w. The DE may be represented as a standard scalar field ϕ minimally coupled to gravity with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \qquad (10)$$

The stress energy-tensor take the form of a perfect fluid represented by

$$T^{\phi}_{\mu\nu} = (\rho_\phi + p_\phi)U_\mu U_\nu g_{\mu\nu} \qquad (11)$$

where $p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)$, $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$.

The equation of state is then $w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{-1+\frac{\dot{\phi}^2}{2V}}{1+\frac{\dot{\phi}^2}{2V}}$ which gives rise to different candidates of scalar field dark energy depending upon the potential $V(\phi)$ of the field ϕ. For slow roll scalar field (potential dominated) i.e. $V(\phi) \gg \dot{\phi}^2$, $w_\phi = -1$ and act like a cosmological constant.

With this set up the EFEs with scalar field in FLRW background yields the following field equations

Friedmann equation

$$H^2 = \frac{1}{3M_{pl}^2}(\rho + \rho_\phi) \qquad (12)$$

And the evolution of the scalar field is governed by the wave equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0 \qquad (13)$$

Where, $M_{pl} = (8\pi G)^{-1/2}$. With this basic set up, we can explore the physical nature and geometrical behaviour of the Universe with certain constraints.

In the current cosmology, it is a common trend for the theoreticians to construct models of the Universe with this basic set up and with certain physical assumptions that solve the system explicitly and cosmic history can be studied from the beginning to present phase of the Universe and eventually the fate. On the contrary, the dynamical system approach can also be used to explore without finding the exact solutions.

I am interested in finding <u>the exact solutions</u> to field equations in a <u>model independent way</u> or the <u>cosmological parametrization</u> and study the various phases of Universe to discuss various phenomena with observation.

COSMOLOGICAL PARAMETERS

Einstein Field Equations is generally characterized by the following basic parameters.

Geometrical Parameters	Definition	Physical Parameters	Definition
Scale factor	a	Energy density	ρ
Hubble parameter	H	Pressure	p
Deceleration parameter	q	EoS parameter	w

Physical Parameters with DE	Definition
Cosmological Constant	Λ
Scalar field	ϕ
Scalar field potential	V

Some other parameters are Density parameter (Ω), Shear (σ) for anisotropic background, scalar expansion (θ), jerk parameter (j) and so on.

COSMOLOGICAL PARAMETRIZATION

- **We have already mentioned earlier, in FLRW cosmology, EFE contains three variables $a(t)$, $\rho(t)$, $p(t)$ with two independent equations that can be solved by supplementing the EoS and system becomes more complicated with the addition of an extra degree of freedom (DE).**
- **In literature, there are several physical arguments to consider a functional form (parametrization) of any cosmological parameter ($a, H, q, j, \rho, p, w, \Omega$ etc.) with some free parameters generally termed as model parameters that can be constrained through any observational datasets.**

- **If we examine closely, we might remark that the primary type of parametrization of geometrical parameters is studied to produce exact solutions that address the expanding dynamics of the universe and give the time evolution of the physical parameters ρ, p, or w. The second type of parametrization of physical parameters is commonly used to explain physical features of the universe.**
- **The model-independent way approach has the potential of rebuilding the cosmic history of the universe as well as interpreting some of the universe's phenomena without affecting the background theory.**

- **This model-independent study of cosmological models termed as <u>COSMOLOGICAL PARAMETRIZATION</u>.**
- **Furthermore, this strategy gives the easiest way to theoretically overcome several problems of standard model, including the initial singularity problem, cosmological constant problem, etc. and also the Hubble tension.**
- **In the following, I have summarized these cosmological parametrizations used in the past 20-40 years in some detail.**

Scale factor $a(t)$

$a(t) = constant$ (Static model)

$a(t) = ct$ (Milne model or Linear expansion)

$a(t) \sim \exp(H_0 t)$ (ΛCDM model or Exponential expansion)

$a(t) \sim \exp\left[-\alpha t \ln\left(\frac{t}{t_0}\right) + \beta t\right]$ (Inflationary model)

$a(t) \sim \exp\left[-\alpha t - \beta t^n\right]$ (Inflationary model)

$a(t) \sim \left[\exp(\alpha t) - \beta \exp(-\alpha t)\right]^n$ (Inflationary model)

$a(t) \sim \exp\left(\frac{t}{M}\right)\left[1 + \cos\left(\frac{\varsigma(t)}{N}\right)\right]$ (quasi steady state cosmology, Cyclic Universe)

$a(t) \sim t^\alpha$ (Power law Cosmology)

$a(t) \sim t^n \exp(\alpha t)$ (Hybrid expansion)

$a(t) \sim \exp\left[n(\log t)^m\right]$ (Logamediate expansion)

$a(t) \sim \cosh \alpha t$ (Hyperbolic expansion)

$a(t) \sim (\sinh \alpha t)^{\frac{1}{n}}$ (Hyperbolic expansion)

$a(t) \sim \left(\frac{t}{t_s - t}\right)^n$ (Singular model)

$a(t) \sim t^n \exp\left[\alpha(t_s - t)\right]$ (Singular model)

$a(t) \sim \exp\left(\alpha \frac{t^2}{t_*^2}\right)$ (Bouncing Model)

$a(t) \sim \exp\left(\frac{\beta}{\alpha+1}(t - t_s)^{\alpha+1}\right)$ (Bouncing Model)

$a(t) \sim \left(\frac{3}{2}\rho_{cr} t^2 + 1\right)^{\frac{1}{3}}$ (Bouncing Model)

$a(t) \sim \left(\frac{t_s - t}{t_*}\right)$ (Bouncing Model)

$a(t) \sim \sin^2\left(\alpha \frac{t}{t_*}\right)$ (Bouncing Model)

Hubble parameter $H(t)$ or $H(a)$

$H(a) = Da^{-m}$

$H(a) = e^{\frac{1-\gamma a^2}{\alpha a}}$

$H(a) = \alpha(1 + a^{-n})$

$H(t) = \frac{m}{\alpha t+\beta}$

$H(t) = \frac{16\alpha t}{15[1+(8\alpha t^2)/5]}$

$H(t) = m + \frac{n}{t}$

$H(t) = \frac{\alpha t_R}{t(t_R-t)}$

$H(t) = \frac{\alpha}{3}(t + T_0)^3 - \beta(t + T_0) + \gamma$

$H(t) = \alpha e^{\lambda t}$

$H(t) = \alpha + \beta(t_s - t)^n$

$H(t) = \alpha - \beta e^{-nt}$

$H(t) = f_1(t) + f_2(t)(t_s - t)^n$

$H(t) = \frac{\beta t^m}{(t^n+\alpha)^p}$

$H(t) = n\alpha \tanh(m - nt) + \beta$

$H(t) = \alpha \tanh\left(\frac{t}{t_0}\right)$

$H(z) = [\alpha + (1 - \alpha)(1 + z)^n]^{\frac{3}{2n}}$

Deceleration parameter $q(t)$ or $q(a)$, $q(z)$

$q(t) = m - 1$

$q(t) = -\alpha t + m - 1$

$q(t) = \alpha \cos(\beta t) - 1$

$q(t) = -\frac{\alpha t}{1+t}$

$q(t) = -\frac{\alpha(1-t)}{1+t}$

$q(t) = -\frac{\alpha}{t^2} + \beta - 1$

$q(t) = (8n^2 - 1) - 12nt + 3t^2$

$q(a) = -1 - \frac{\alpha a^\alpha}{1+a^\alpha}$

$q(z) = q_0 + q_1 z$

$q(z) = q_0 + q_1 z(1 + z)^{-1}$

$q(z) = q_0 + q_1 z(1 + z)(1 + z^2)^{-1}$

$q(z) = \frac{1}{2} + q_1(1 + z)^{-2}$

$q(z) = q_0 + q_1[1 + \ln(1 + z)]^{-1}$

$q(z) = \frac{1}{2} + (q_1 z + q_2)(1 + z)^{-2}$

$$q(z) = -1 + \frac{3}{2}\left(\frac{(1+z)^{q_2}}{q_1+(1+z)^{q_2}}\right)$$
$$q(z) = -\frac{1}{4}\left[3q_1 + 1 - 3(q_1+1)\left(\frac{q_1 e^{q_2(1+z)} - e^{-q_2(1+z)}}{q_1 e^{q_2(1+z)} + e^{-q_2(1+z)}}\right)\right]$$
$$q(z) = -\frac{1}{4} + \frac{3}{4}\left(\frac{q_1 e^{q_2 \frac{z}{\sqrt{1+z}}} - e^{-q_2 \frac{z}{\sqrt{1+z}}}}{q_1 e^{q_2 \frac{z}{\sqrt{1+z}}} + e^{-q_2 \frac{z}{\sqrt{1+z}}}}\right)$$
$$q(z) = q_f + \frac{q_i - q_f}{1 - \frac{q_i}{q_f}\left(\frac{1+z_t}{1+z}\right)^{\frac{1}{\tau}}}$$
$$q(z) = q_0 - q_1\left(\frac{(1+z)^{-\alpha}-1}{\alpha}\right)$$
$$q(z) = q_0 + q_1\left[\frac{\ln(\alpha+z)}{1+z} - \beta\right]$$
$$q(z) = q_0 - (q_0 - q_1)(1+z)\exp\left[z_c^2 - (z+z_c)^2\right]$$

Jerk parameter $j(z)$

$j(z) = -1 + j_1 \frac{f(z)}{E^2(z)}$, where $f(z) = z$, $\frac{z}{1+z}$, $\frac{z}{(1+z)^2}$, $\log(1+z)$ and $E(z) = \frac{H(z)}{H_0}$

$j(z) = -1 + j_1 \frac{f(z)}{h^2(z)}$, where $f(z) = 1$, $1+z$, $(1+z)^2$, $(1+z)^{-1}$ and $h(z) = \frac{H(z)}{H_0}$

Pressure $p(\rho)$, $p(z)$

$p(\rho) = w\rho$ (Perfect fluid EoS)

$p(\rho) = w\rho - f(H)$ (Viscous fluid EoS)

$p(\rho) = w\rho + k\rho^{1+\frac{1}{n}}$ (Polytropic gas EoS)

$p(\rho) = \frac{8w\rho}{3-\rho} - 3\rho^2$ (Vanderwaal gas EoS)

$p(\rho) = -(w+1)\frac{\rho^2}{\rho_P} + w\rho + (w+1)\rho_\Lambda$ (EoS in quadratic form)

$p(\rho) = -\frac{B}{\rho}$ (Chaplygin gas EoS)

$p(\rho) = -\frac{B}{\rho^\alpha}$ (Generalized Chaplygin gas EoS)

$p(\rho) = A\rho - \frac{B}{\rho^\alpha}$ (Modified Chaplygin gas EoS)

$p(\rho) = A\rho - \frac{B(a)}{\rho^\alpha}$ (Variable modified Chaplygin gas EoS)

$p(\rho) = A(a)\rho - \frac{B(a)}{\rho^\alpha}$ (New variable modified Chaplygin gas EoS)

$p(\rho) = -\rho - \rho^\alpha$ (DE EoS)

$p(z) = \alpha + \beta z$ (DE EoS)

$p(z) = \alpha + \beta\frac{z}{1+z}$ (DE EoS)

$p(z) = \alpha + \beta\left(z + \frac{z}{1+z}\right)$ (DE EoS)

$p(z) = \alpha + \beta\ln(1+z)$ (DE EoS)

Equation of state parameter $w(z)$

$w(z) = w_0 + w_1 z$ (Linear parametrization)

$w(z) = w_0 + w_1 \frac{z}{(1+z)^2}$ (JBP parametrization)

$w(z) = w_0 + w_1 \frac{z}{(1+z)^n}$ (Generalized JBP parametrization)

$w(z) = w_0 + w_1 \frac{z}{1+z}$ (CPL parametrization)

$w(z) = w_0 + w_1 \left(\frac{z}{1+z}\right)^n$ (Generalized CPL parametrization)

$w(z) = w_0 + w_1 \frac{z}{\sqrt{1+z^2}}$ (Square-root parametrization)

$w(z) = w_0 + w_1 \sin(z)$ (Sine parametrization)

$w(z) = w_0 + w_1 \ln(1 + z)$ (Logarithmic parametrization)

$w(z) = w_0 + w_1 \ln\left(1 + \frac{z}{1+z}\right)$ (Logarithmic parametrization)

$w(z) = w_0 + w_1 \frac{z(1+z)}{1+z^2}$ (BA parametrization)

$w(z) = w_0 + w_1 \left(\frac{\ln(2+z)}{1+z} - \ln 2\right)$ (MZ parametrization)

$w(z) = w_0 + w_1 \left(\frac{\sin(1+z)}{1+z} - \sin 1\right)$ (MZ parametrization)

$w(z) = w_0 + w_1 \frac{z}{1+z^2}$ (FSLL parametrization)

$w(z) = w_0 + w_1 \frac{z^2}{1+z^2}$ (FSLL parametrization)

$w(z) = -1 + \frac{1+z}{3} \frac{\alpha+2\beta(1+z)}{\gamma+2\alpha(1+z)+\beta(1+z)^2}$ (ASSS parametrization)

$w(z) = \frac{1+\left(\frac{1+z}{1+z_s}\right)^\alpha}{w_0+w_1\left(\frac{1+z}{1+z_s}\right)^\alpha}$ (Hannestad Mortsell parametrization)

$w(z) = -1 + \alpha(1 + z) + \beta(1 + z)^2$ (Polynomial parametrization)

$w(z) = -1 + \alpha\,[1 + f(z)] + \beta\,[1 + f(z)]^2$ (Generalized Polynomial parametrization)

$$w(z) = w_0 + z\left(\frac{dw}{dz}\right)_0$$
$$w(z) = \frac{-2(1+z)d_c'' - 3d_c'}{3\left[d_c' - \Omega_M(1+z)^3(d_c')^3\right]} \text{ where } d_c' = \int_0^z \frac{H_0 dz}{H(z)}$$
$$w_x(a) = w_0 \exp(a-1)$$
$$w_x(a) = w_0 a(1 - \log a)$$
$$w_x(a) = w_0 a \exp(1-a)$$
$$w_x(a) = w_0 a(1 + \sin(1-a))$$
$$w_x(a) = w_0 a(1 + \arcsin(1-a))$$
$$w_{de}(z) = w_0 + w_1 q$$
$$w_{de}(z) = w_0 + w_1 q(1+z)^\alpha$$
$$w_{de}(z) = \frac{w_0}{[1 + b\ln(1+z)]^2}$$
$$w_x(z) = w_0 + b\{1 - \cos[\ln(1+z)]\}$$
$$w_x(z) = w_0 + b\sin[\ln(1+z)]$$
$$w_x(z) = w_0 + b\left[\frac{\sin(1+z)}{1+z} - \sin 1\right]$$
$$w_x(z) = w_0 + b\left(\frac{z}{1+z}\right)\cos(1+z)$$
$$w(z) = w_0 + w_a\left[\frac{\ln(2+z)}{1+z} - \ln 2\right]$$
$$w(z) = w_0 + w_a\left[\frac{\ln(\alpha+1+z)}{\alpha+z} - \frac{\ln(\alpha+1)}{\alpha}\right]$$

Energy density ρ

$$\rho = \rho_c$$
$$\rho \sim \theta^2$$
$$\rho = \frac{A}{a^4}\sqrt{a^2+b}$$
$$(\rho + 3p)\, a^3 = A$$
$$\rho + p = \rho_c$$
$$\rho_{de}(z) = \rho_{de}(0)\left[1 + \alpha\left(\frac{z}{1+z}\right)^n\right]$$
$$\rho_{de}(z) = \frac{1}{\rho_\phi}\left(\frac{d\rho_\phi}{d\phi}\right) = -\frac{\alpha a}{(\beta+a)^2}$$
$$\rho_{de}(z) = \alpha H(z)$$
$$\rho_{de}(z) = \alpha H(z) + \beta H^2(z)$$
$$\rho_{de}(z) = \frac{3}{\kappa^2}\left[\alpha + \beta H^2(z)\right]$$
$$\rho_{de}(z) = \frac{3}{\kappa^2}\left[\alpha + \frac{2}{3}\beta \dot{H}(z)\right]$$
$$\rho_{de}(z) = \frac{3}{\kappa^2}\left[\alpha H^2(z) + \frac{2}{3}\beta \dot{H}(z)\right]$$
$$\rho_{de}(z) = \frac{3}{\kappa^2}\left[\alpha + \beta H^2(z) + \frac{2}{3}\gamma \dot{H}(z)\right]$$
$$\rho_{de}(z) = \rho_{\phi 0}(1+z)^\alpha e^{\beta z}$$

Scalar field Potentials $V(\phi)$

$V(\phi) = V_0\phi^n$ (Power law)

$V(\phi) = V_0 \exp\left[-\frac{\alpha\phi}{M_{pl}}\right]$ (exponential)

$V(\phi) = \frac{V_0}{\cosh[\phi/\phi_0]}$

$V(\phi) = V_0\left[\cosh\left(\alpha\phi/M_{pl}\right)\right]^{-\beta}$ (hyperbolic)

$V(\phi) = \frac{\alpha}{\phi^n}$ (Inverse power law)

$V(\phi) = \frac{V_0}{1+\beta\exp(-\alpha\kappa\phi)}$ (Woods-Saxon potential)

$V(\phi) = \alpha c^2\left[\tanh\frac{\phi}{\sqrt{6}\alpha}\right]^2$ (α-attractor)

$V(\phi) = V_0(1+\phi^\alpha)^2$

$V(\phi) = V_0\exp(\alpha\phi^2)$

$V(\phi) = \frac{1}{4}(\phi^2-1)^2$

Cosmological constant (Λ)

$\Lambda \sim a^{-n}$

$\Lambda \sim H^n$

$\Lambda \sim \rho$

$\Lambda \sim t^n$

$\Lambda \sim q^n$

$\Lambda \sim e^{-\beta a}$

$\Lambda = \Lambda(T)$ T is Temperature

$\Lambda \sim C + e^{-\beta t}$

$\Lambda = 3\beta H^2 + \alpha a^{-2}$

$\Lambda = \beta\frac{\ddot{a}}{a}$

$\Lambda = 3\beta H^2 + \alpha\frac{\ddot{a}}{a}$

$\frac{d\Lambda}{dt} \sim \beta\Lambda - \Lambda^2$

Note: All the parametrization listed above contain some arbitrary constants such as α, β, γ, m, n, p, q_0, q_1, q_2, w_0, w_1, A, B are model parameters which are generally constrained through observational datasets or through any analytical methods and also some arbitrary functions $f_1(t)$, $f_2(t)$. t_s denote the bouncing time or future singularity time and t_* some arbitrary time.

EXEMPLIFICATION

FIELD EQUATIONS WITH QUINTESSENCE

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}M_{Pl}^2 R + L_m + L_\phi\right)$$

$$R_{\mu\nu} = -\frac{1}{2}Rg_{\mu\nu} = M_{Pl}^{-2}T_{\mu\nu}^{Total},$$ where $$T_{\mu\nu}^{Total} = T_{\mu\nu}^{M} + T_{\mu\nu}^{\phi}.$$

$$T_{\mu\nu}^{\phi} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\partial\phi\right)^2 - g_{\mu\nu}V\left(\phi\right),$$ Here, $$\left(\partial\phi\right)^2 \equiv g_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi$$

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

$$M_{Pl}^{-2}\rho_{Total} = 3\left(\frac{\dot{a}}{a}\right)^2 = 3H^2, \quad M_{Pl}^{-2}p_{Total} = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = (2q-1)H^2.$$

$\rho_{Total} = \rho_{eff} = \rho + \rho_\phi$ and $p_{Total} = p_{eff} = p + p_\phi$.

$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. $\quad \omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$.

SOLUTION OF FE IN A MODEL-INDEPENDENT WAY

Taylor series expansion of a(t) in the vicinity of the present time $t = t_0$ is,

$$a(t) = a\left(t_0\right) + \dot{a}\left(t_0\right)\left[t - t_0\right] + \frac{1}{2}\ddot{a}\left(t_0\right)\left[t - t_0\right]^2 + \cdots .$$

$$\frac{a(t)}{a\left(t_0\right)} = 1 + H_0\left[t - t_0\right] - \frac{q_0}{2}H_0^2[t - t_0]^2 + \cdots , \qquad H(t) = \frac{\dot{a}}{a} , q(t) = -\frac{a\ddot{a}}{\dot{a}^2}.$$

$a(t) = Ce^{\int H(t)dt}$, where C is a constant of integration.

$$q(t) = -1 + \frac{d}{dt}\left(\frac{1}{H(t)}\right),$$

Various other relations can also be established among these cosmological parameters.

Hubble parameter

$$H(t) = +\frac{1}{a}\frac{da}{dt}$$

Deceleration parameter

$$q(t) = -\frac{1}{a}\frac{d^2a}{dt^2}\left[\frac{1}{a}\frac{da}{dt}\right]^{-2}$$

Jerk parameter

$$j(t) = +\frac{1}{a}\frac{d^3a}{dt^3}\left[\frac{1}{a}\frac{da}{dt}\right]^{-3}$$

snap parameter

$$s(t) = +\frac{1}{a}\frac{d^4a}{dt^4}\left[\frac{1}{a}\frac{da}{dt}\right]^{-4}$$

Lerk parameter

$$l(t) = +\frac{1}{a}\frac{d^5a}{dt^5}\left[\frac{1}{a}\frac{da}{dt}\right]^{-5}$$

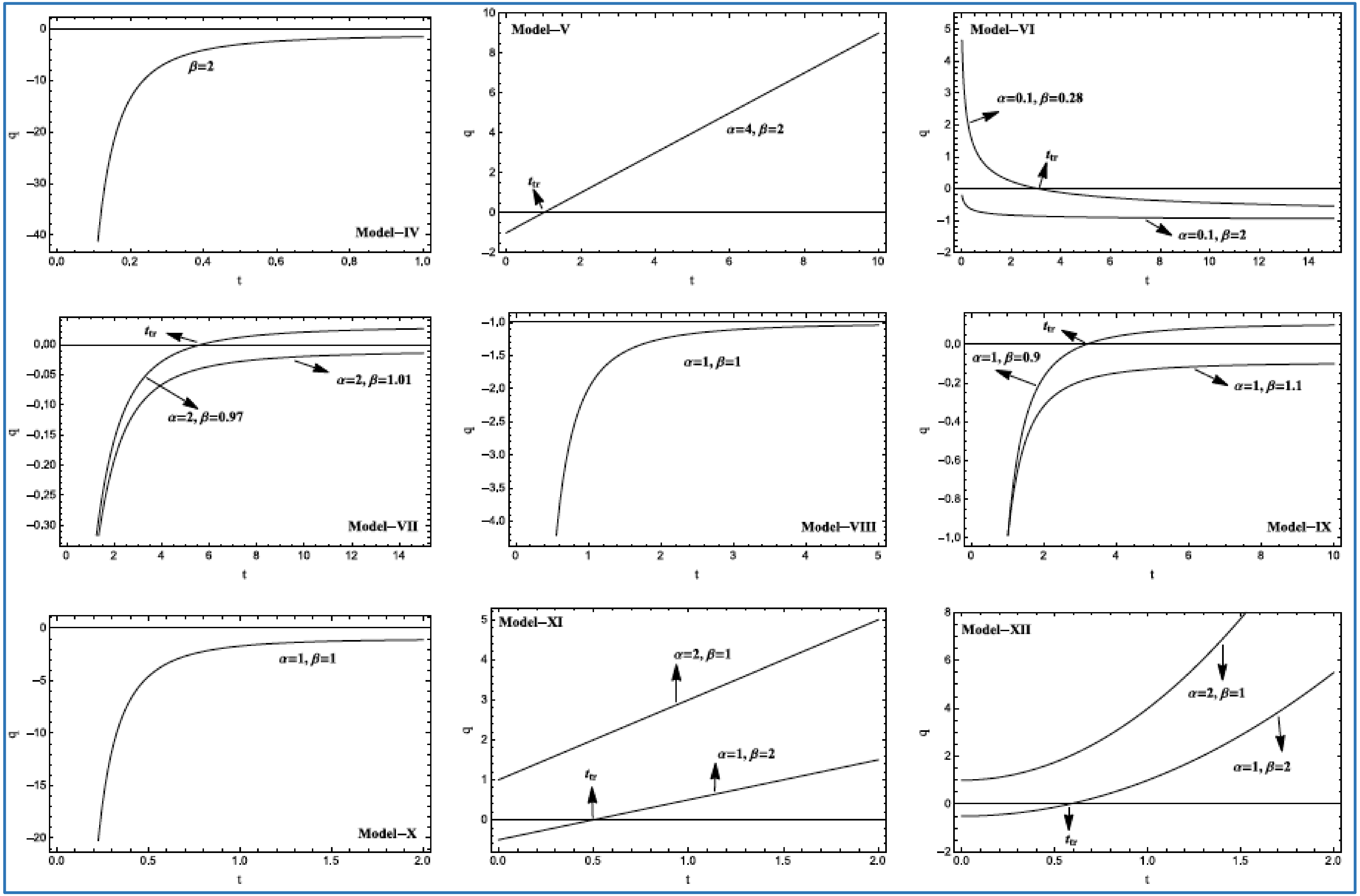
Motivated by the discussion, we consider the parametrization of H as

$$\boxed{H(t) = \frac{\beta t^m}{(t^n + \alpha)^p}}$$

where $\alpha, \beta \neq 0, m, n, p$ are real constants (better call them model parameters). α and β both have the dimensions of time. The specific values of m, n, p will suggest the different forms of HP and produce interesting cosmologies. Our parametrization generalizes several known models which were obtained by the parametrization of any cosmological parameters $a(t), H(t), q(t), \Lambda(t), \rho(t)$ or $w(t)$ in different contexts.

- This one parametrization covers many models obtained in the past few decades in different schemes of parametrization under one umbrella.
- This simple modification in the functional form of HP can give rise to interesting cosmological phenomena such as big rip singularity, bounce and others. Few models admit transition from deceleration to acceleration as suggested by some observations.

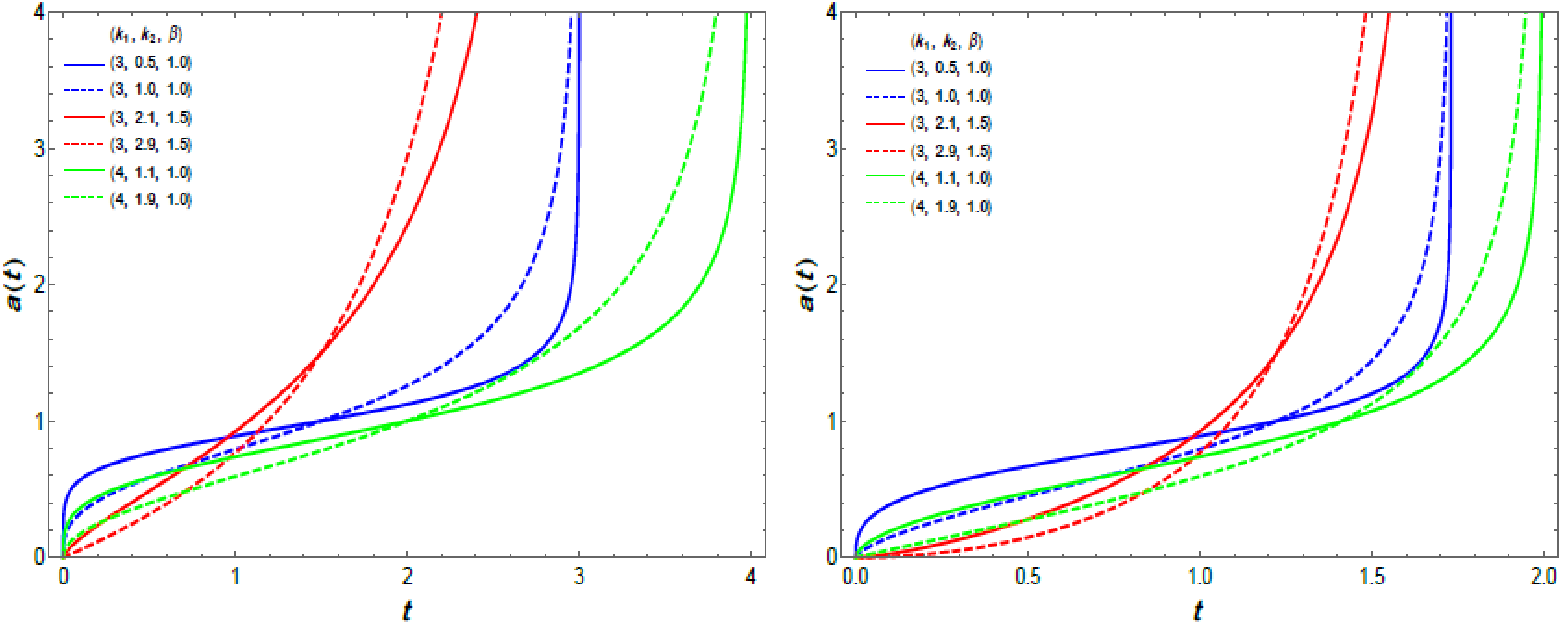
Models	Specific values of m, n, p	HP $H(t)$	SF $a(t)$	DP $q(t)$
I	$m = 0, p = 0, \forall\, n$	β	$Ce^{\beta t}$	-1
II	$m = -1, p = 0, \forall\, n$	$\frac{\beta}{t}$	Ct^{β}	$-1 + \frac{1}{\beta}$
III	$m = 0, p = 1, n = 1$	$\frac{\beta}{t+\alpha}$	$C(t+\alpha)^{\beta}$	$-1 + \frac{1}{\beta}$
IV	$m = 1, p = 0, \forall\, n$	βt	$Ce^{\beta \frac{t^2}{2}}$	$-1 - \frac{1}{\beta}\frac{1}{t^2}$
V	$m = 0, p = 1, n = 2$	$\frac{\beta}{t^2+\alpha}$	$Ce^{\frac{\beta}{\sqrt{\alpha}} \tan^{-1} \frac{t}{\sqrt{\alpha}}}$	$-1 + \frac{2}{\beta}t$
VI	$m = 0, p = \frac{1}{2}, n = 1$	$\frac{\beta}{\sqrt{t+\alpha}}$	$Ce^{2\beta\sqrt{t+\alpha}}$	$-1 + \frac{1}{2\beta}\frac{1}{\sqrt{t+\alpha}}$
VII	$m = 0, p = \frac{1}{2}, n = 2$	$\frac{\beta}{\sqrt{t^2+\alpha}}$	$C(t+\sqrt{t^2+\alpha})^{\beta}$	$-1 + \frac{1}{\beta}\frac{t}{\sqrt{t^2+\alpha}}$
VIII	$m = 1, p = 1, n = 1$	$\frac{\beta t}{t+\alpha}$	$Ce^{\beta t}(t+\alpha)^{-\alpha\beta}$	$-1 - \frac{\alpha}{\beta}\frac{1}{t^2}$
IX	$m = 1, p = 1, n = 2$	$\frac{\beta t}{t^2+\alpha}$	$C(t^2+\alpha)^{\frac{\beta}{2}}$	$-1 + \frac{1}{\beta} - \frac{\alpha}{\beta}\frac{1}{t^2}$
X	$m = 1, p = \frac{1}{2}, n = 2$	$\frac{\beta t}{\sqrt{t^2+\alpha}}$	$Ce^{\beta\sqrt{t^2+\alpha}}$	$-1 - \frac{\alpha}{\beta}\frac{1}{t^2\sqrt{t^2+\alpha}}$
XI	$m = -1, p = 1, n = 1$	$\frac{\beta}{t(t+\alpha)}$	$C(\frac{t}{t+\alpha})^{\frac{\beta}{\alpha}}$	$-1 + \frac{\alpha}{\beta} + \frac{2}{\beta}t$
XII	$m = -1, p = 1, n = 2$	$\frac{\beta}{t(t^2+\alpha)}$	$C(\frac{t^2}{t^2+\alpha})^{\frac{\beta}{2\alpha}}$	$-1 + \frac{\alpha}{\beta} + \frac{3}{\beta}t^2$



Plots of $q(t) \sim t$ for models obtained. Some models show eternal acceleration and some show deceleration-acceleration phase transition. Some show acceleration-deceleration transition.

For negative α and β, two models show DEC-ACC phase transition and the HP can be expressed in terms of redshift as

Models	$H(t)$	$a(t)$	$q(t)$
M1	$\frac{k_2}{t(k_1-t)}$	$\beta\left(\frac{t}{k_1-t}\right)^{\frac{k_2}{k_1}}$	$-1+\frac{k_1}{k_2}-\frac{2}{k_2}t$
M2	$\frac{k_2}{t(k_1-t^2)}$	$\beta\left(\frac{t^2}{k_1-t^2}\right)^{\frac{k_2}{2k_1}}$	$-1+\frac{k_1}{k_2}-\frac{3}{k_2}t^2$



In order to understand the late time behavior of the Universe, it is convenient to express all the cosmological parameters in terms of redshift. (Redshift may be characterized by the relative difference between the observed and emitted wavelengths of an object and is related to scale factor by, $1+z=\frac{\lambda_{now}}{\lambda_{then}}=\frac{a_{now}}{a_{then}}=\frac{a_0}{a}$.)

The $t-z$ relationship can be established for the discussed models M1 and M2 respectively as,

$$t(z)=k_1\left[1+\{\beta(1+z)\}^{\frac{k_1}{k_2}}\right]^{-1},$$

and

$$t(z)=\sqrt{k_1}\left[1+\{\beta(1+z)\}^{2\frac{k_1}{k_2}}\right]^{-\frac{1}{2}}$$

Now, the models M1 and M2 are described as,

$$H(z)=H_0(1+\beta^{\alpha})^{-2}(1+z)^{-\alpha}\left[1+\{\beta(1+z)\}^{\alpha}\right]^2 \quad \textbf{Model M1}$$

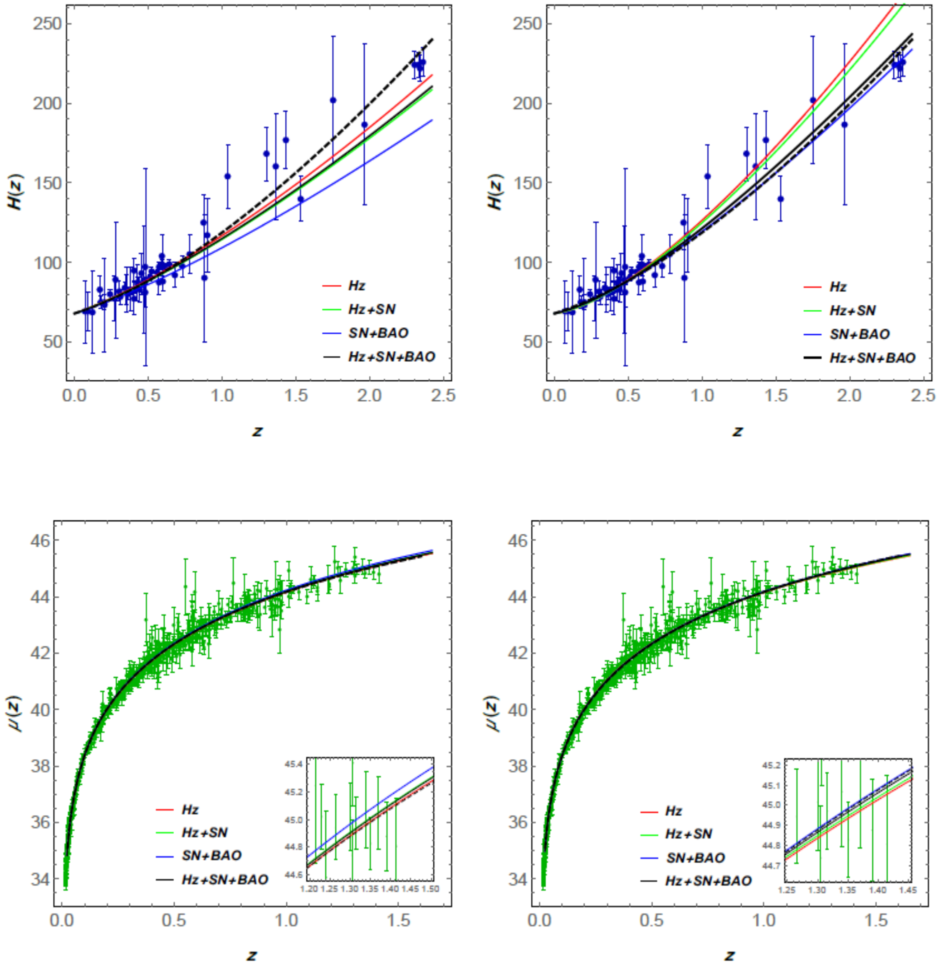
$$H(z)=H_0(1+\beta^{2\alpha})^{-\frac{3}{2}}(1+z)^{-2\alpha}\left[1+\{\beta(1+z)\}^{2\alpha}\right]^{\frac{3}{2}} \quad \textbf{Model M2}$$

The two models can now be compared with some observational datasets. Also, the model parameters α & β are to be estimated through datasets.

Three datasets are considered here, namely Hubble datasets (*Hz*), Type Ia supernovae datasets (*SN*) and Baryon Acoustic Oscillations datasets (*BAO*). Using some statistical techniques, the model parameters α & β are estimated as follows,

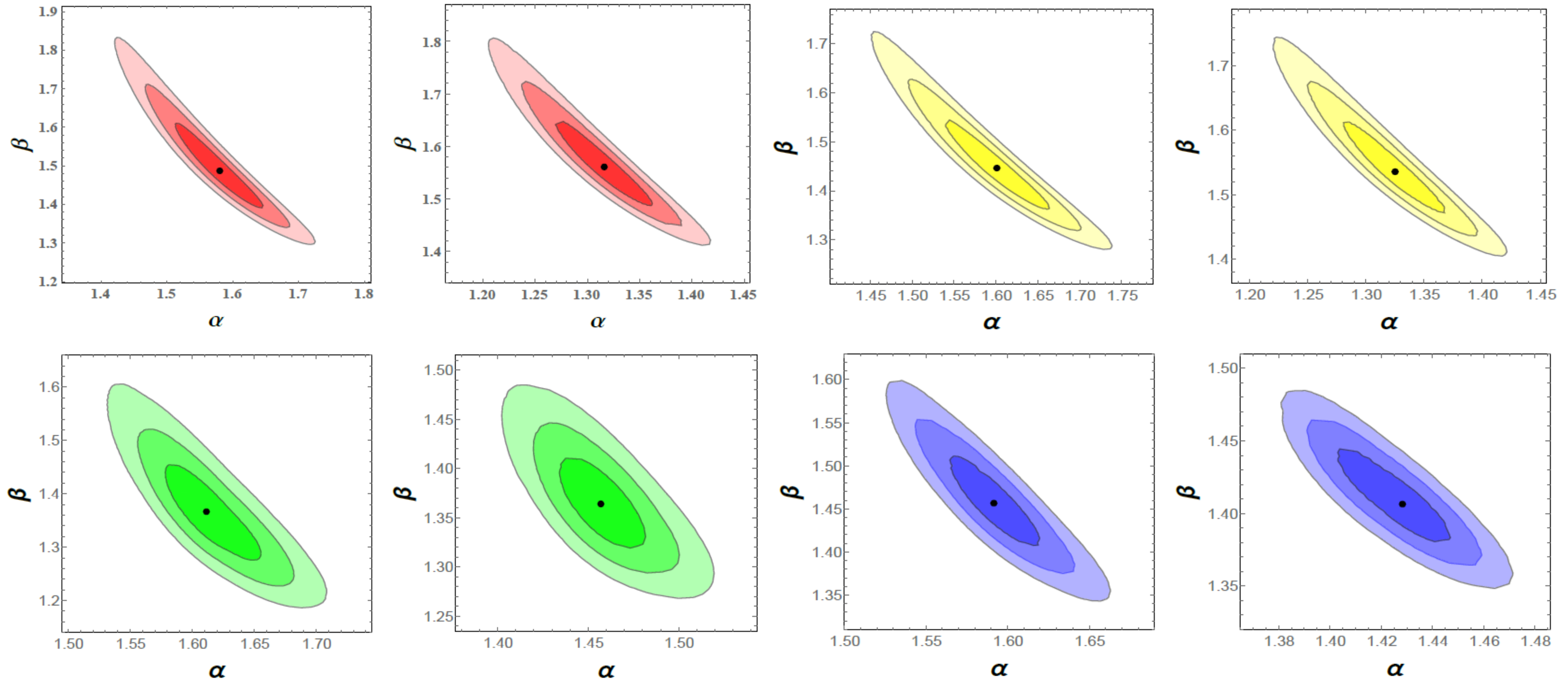
Table-4: Constrained values of model parameters and chi square values

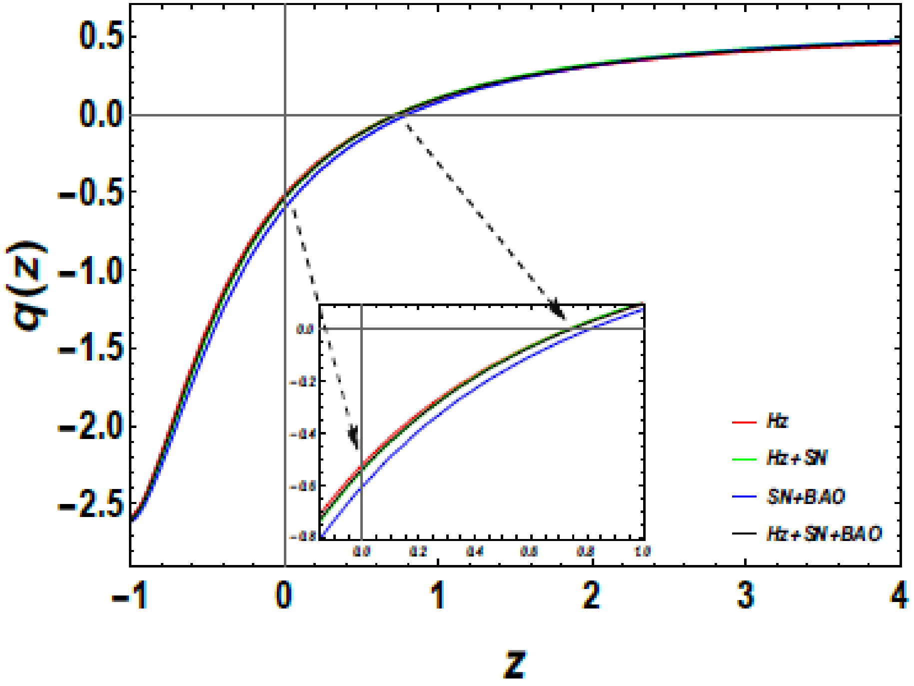
Datasets	Models	α	β	$\chi^2_{\min}$	χ^2/dof
$H(z)$	M1	1.58064	1.48729	31.329529	0.56962
	M2	1.31611	1.56124	29.972660	0.54495
$H(z)+SN$	M1	1.60094	1.44572	596.49325	0.93935
	M2	1.32551	1.53587	595.02853	0.93705
$SN+BAO$	M1	1.61116	1.36647	564.45777	0.96653
	M2	1.45677	1.36451	566.44641	0.96994
$H(z)+SN+BAO$	M1	1.59173	1.45678	599.07805	0.93459
	M2	1.42829	1.40637	614.40132	0.95850

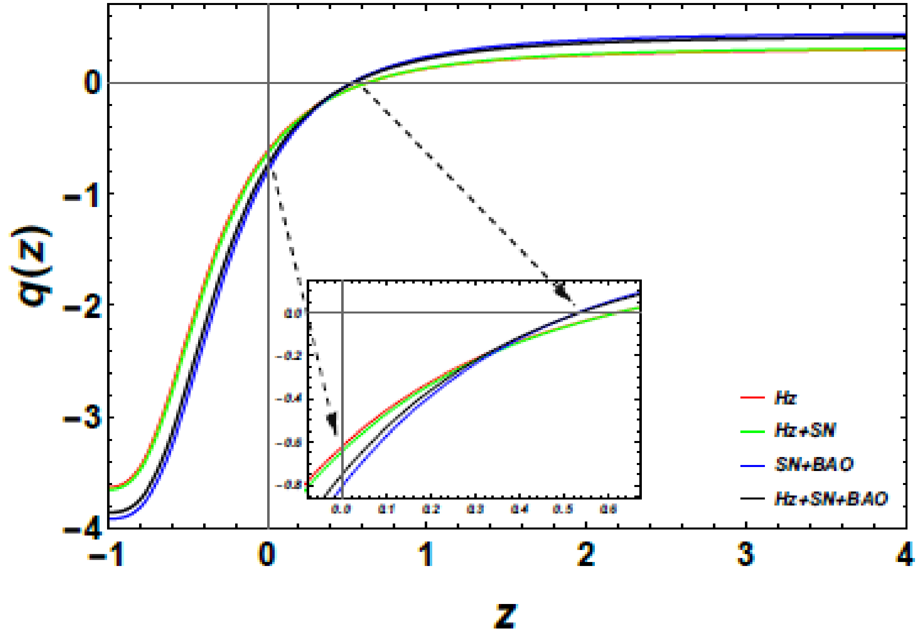


Using these values, we can see the fitting of our models M1 and M2 with the datasets and compare with the standard ΛCDM model.

The maximum likelihood contours for the model parameters α & β are shown in the following figures for independent *Hz* datasets and combined *Hz*+*SN*, *SN*+*BAO* and *Hz*+*SN*+*BAO* datasets respectively with $1\sigma, 2\sigma$ *and* 3σ error contours in the $\alpha - \beta$ plane.







The evolution of the deceleration parameter can be seen from the following figure with the numerical values of the mode parameters for both the models.

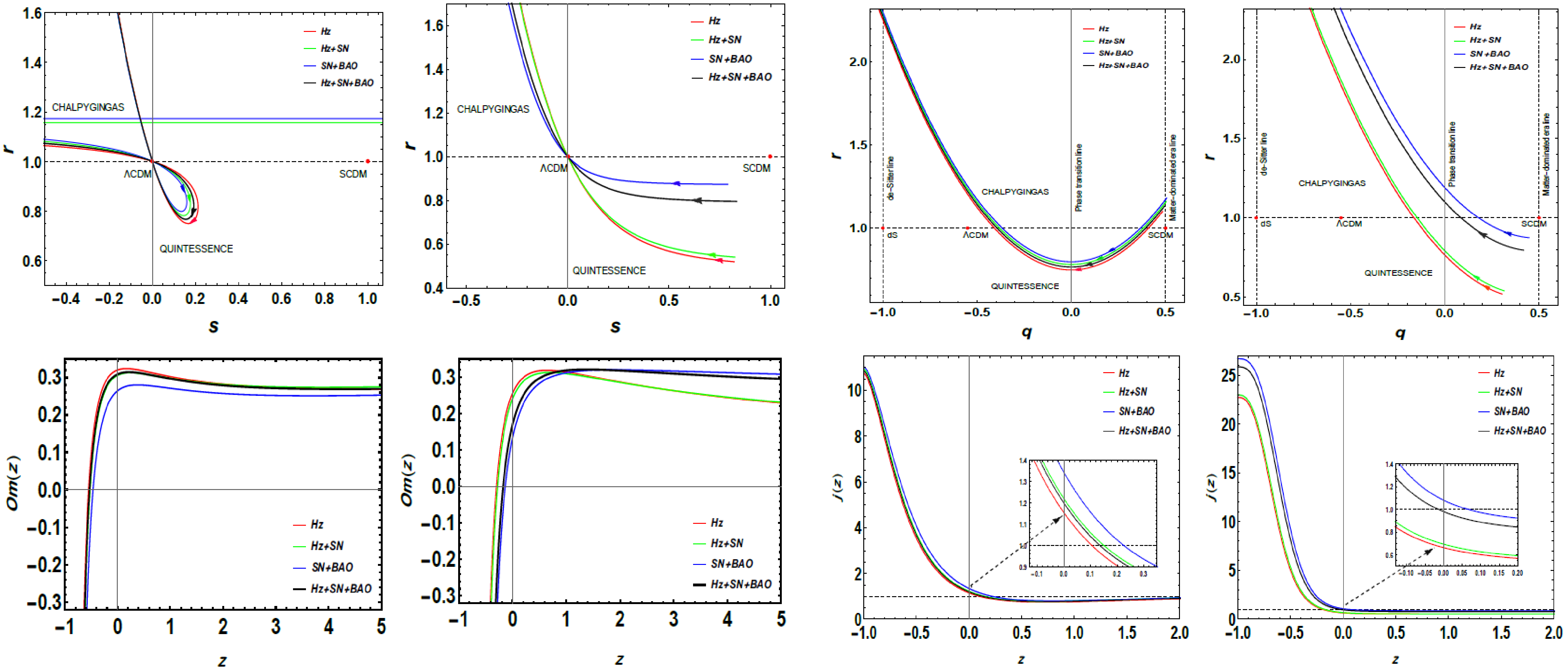
Table-5: Values of q at different epochs & phase transition redshift for model M1

redshift	formula	Hz	$Hz+SN$	$SN+BAO$	$Hz+SN+BAO$
$z\longrightarrow\infty\ (q_i)$	$q_i=-1+\alpha$	0.58064	0.60094	0.61116	0.59173
$z\longrightarrow 0\ (q_0)$	$q_0=-1+\alpha-\frac{2\alpha}{1+\beta^{\alpha}}$	−0.51977	−0.54087	−0.60308	−0.53714
$z\longrightarrow -1\ (q_f)$	$q_f=-1-\alpha$	−2.58064	−2.60094	−2.61116	−2.59173
$z_{tr}\ (q=0)$	$z_{tr}=-1+\frac{1}{\beta}\left(\frac{\alpha+1}{\alpha-1}\right)^{\frac{1}{\alpha}}$	0.72763	0.72730	0.80236	0.73622

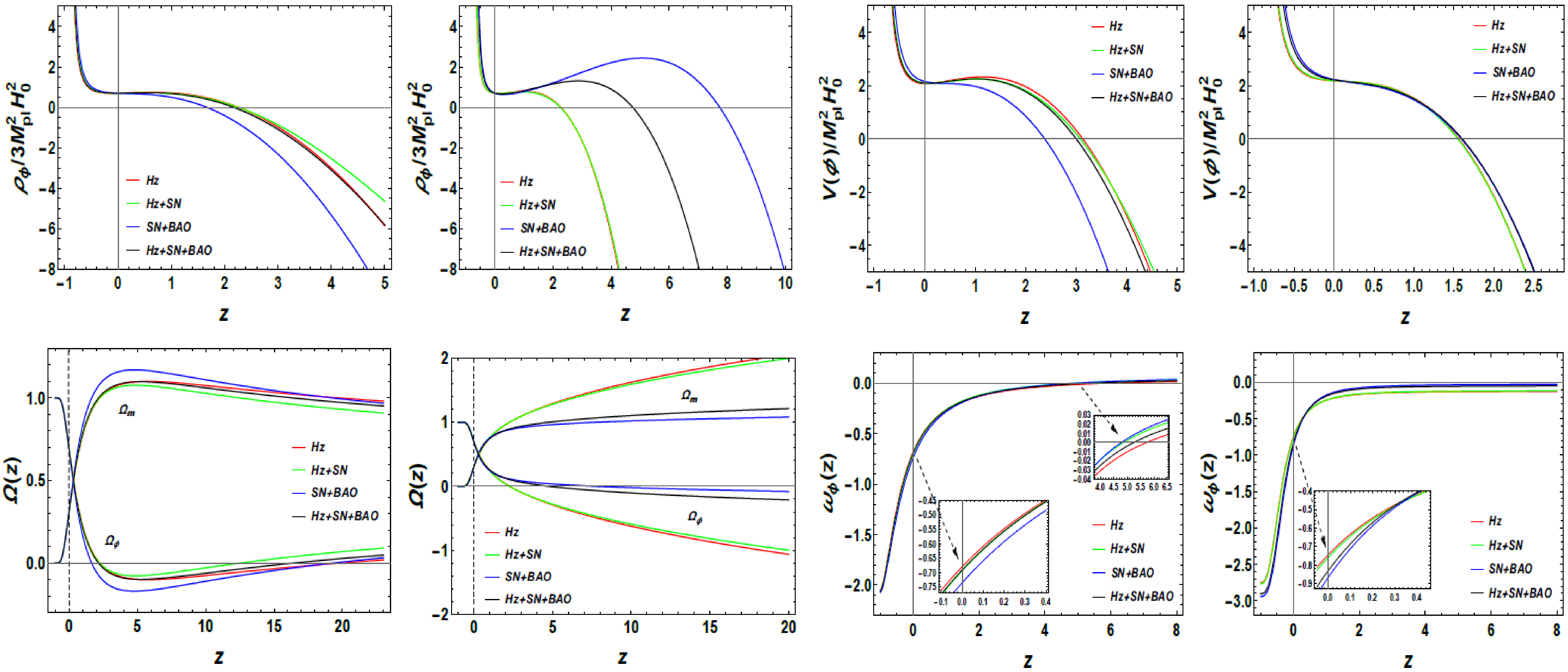
Table-6: Values of q at different epochs & phase transition redshift for model M2

redshift	formula	Hz	$Hz+SN$	$SN+BAO$	$Hz+SN+BAO$
$z\longrightarrow\infty\ (q_i)$	$q_i=-1+\alpha$	0.31611	0.32551	0.45677	0.42829
$z\longrightarrow 0\ (q_0)$	$q_0=-1+\alpha-\frac{3\alpha}{1+\beta^{2\alpha}}$	−0.61721	−0.63987	−0.80152	−0.74601
$z\longrightarrow -1\ (q_f)$	$q_f=-1-2\alpha$	−3.63222	−3.65102	−3.91354	−3.85658
$z_{tr}\ (q=0)$	$z_{tr}=-1+\frac{1}{\beta}\left(\frac{2\alpha+1}{\alpha-1}\right)^{\frac{1}{2\alpha}}$	0.61941	0.62055	0.53182	0.53471

Statefinder diagnostics and Om diagnostic analysis are used to distinguish various dark energy models. The following plots show a comparison of our models with some standard models.



Finally, one can calculate the physical parameters (ρ_ϕ, $V(\phi)$, ω_ϕ, Ω) for the considered models M1 and M2. The behavior is shown in the following figures.



The age of the Universe in both the models is calculated and shown in the following table.

Table-7						
Models	M1			M2		
Datasets	(α, β)	Factor	Age (in Gyr)	(α, β)	Factor	Age (in Gyr)
Hz	$(1.58064, 1.48729)$	0.97046	14.0068	$(1.31611, 1.56124)$	0.99502	14.3613
$Hz + SN$	$(1.60094, 1.44572)$	0.97084	14.0123	$(1.32551, 1.53587)$	0.99630	14.3797
$SN + BAO$	$(1.61116, 1.36647)$	0.99597	14.3751	$(1.45677, 1.36451)$	0.96400	13.9136
$Hz + SN + BAO$	$(1.59173, 1.45678)$	0.97343	14.0496	$(1.42829, 1.40637)$	0.96445	13.9201

Recently, we have discussed this model with improved datasets such as Pantheon, updated BAO and CMB datasets and the results are improved.

Using the constrained values of the model parameters, we have extended the analysis in modified f(Q,T) gravity and performed some more cosmological tests.

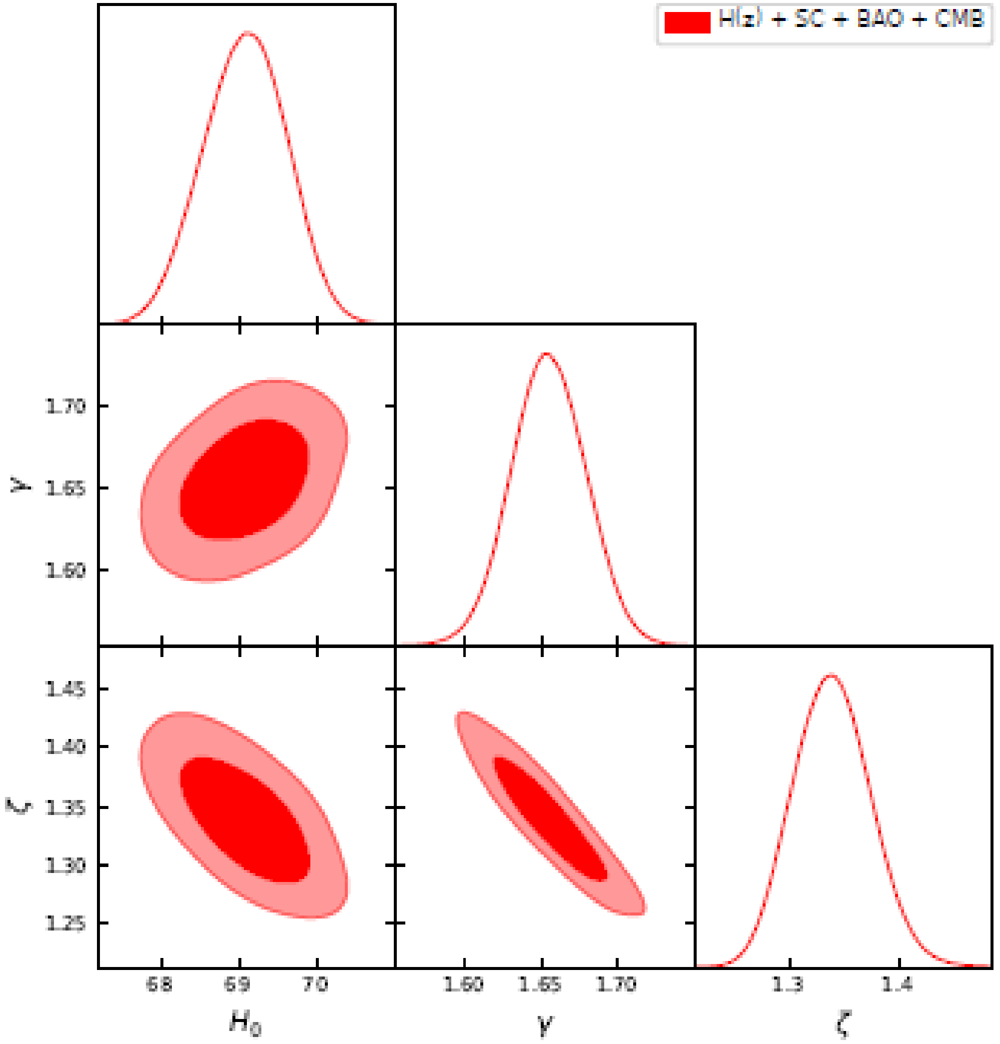


FIG. 1. MCMC confidence contours at 1σ and 2σ for Model 1.

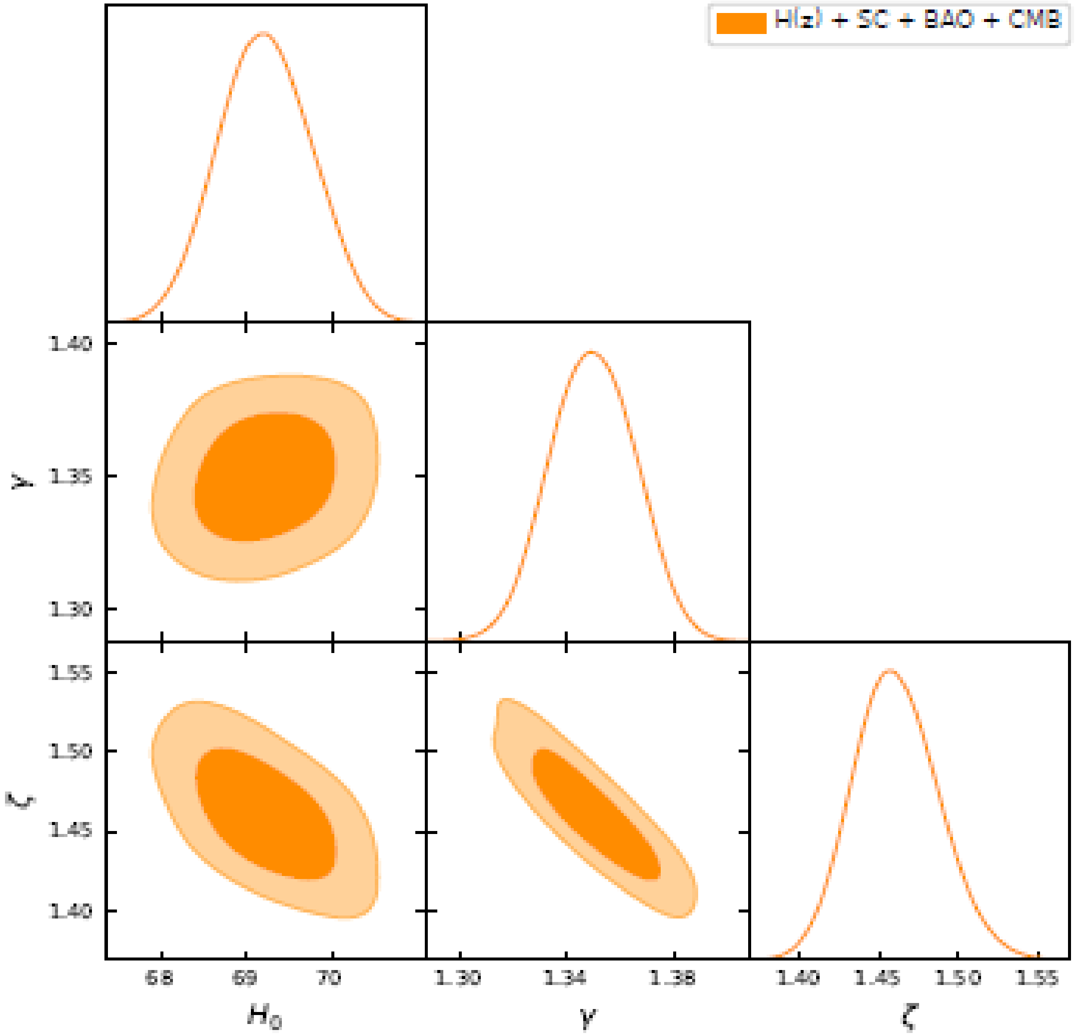
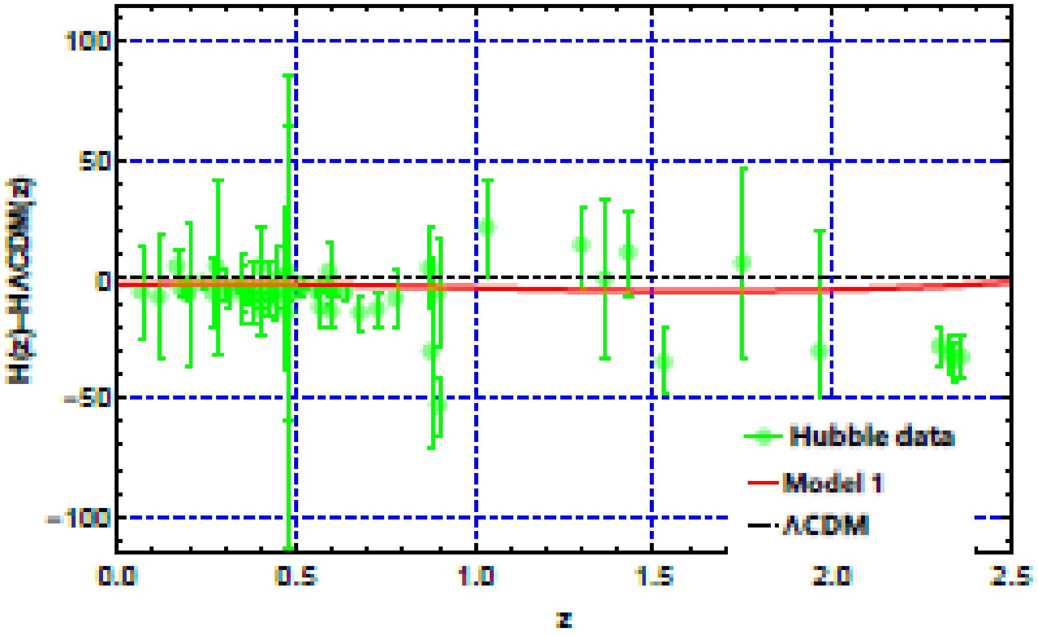
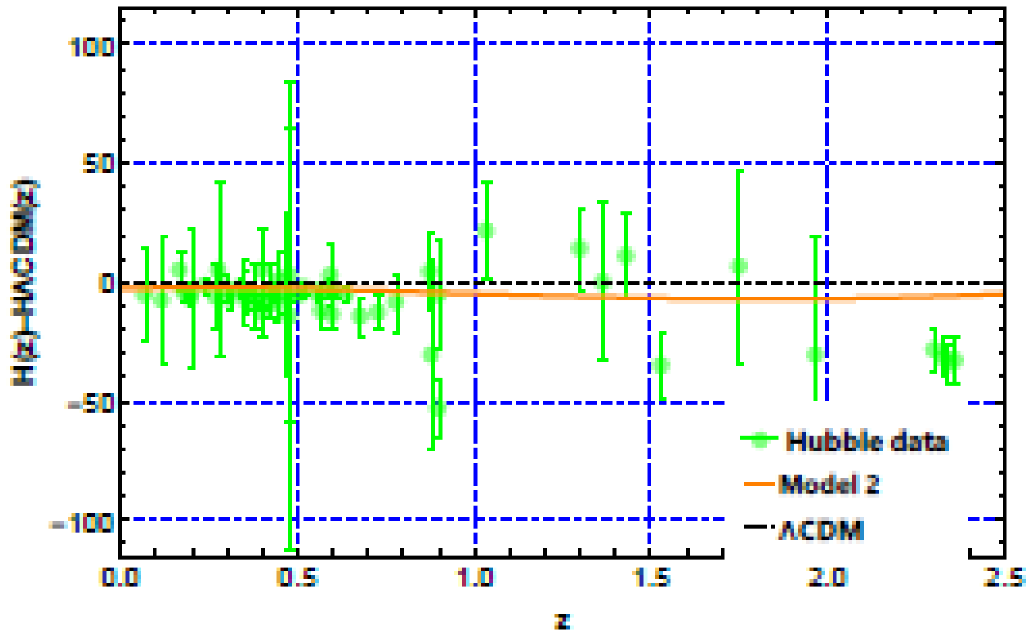


FIG. 2. MCMC confidence contours at 1σ and 2σ for Model 2.

MCMC Results		
Model	Parameters	Bestfit Value
ΛCDM Model	H_0	$69.854848^{+1.259100}_{-1.259100}$
Model 1	H_0	$69.302413^{+0.443452}_{-0.443452}$
	γ	$1.662642^{+0.023615}_{-0.023615}$
	ζ	$1.322252^{+0.027842}_{-0.027842}$
Model 2	H_0	$69.247921^{+0.465229}_{-0.465229}$
	γ	$1.350387^{+0.015168}_{-0.015168}$
	ζ	$1.459203^{+0.024204}_{-0.024204}$

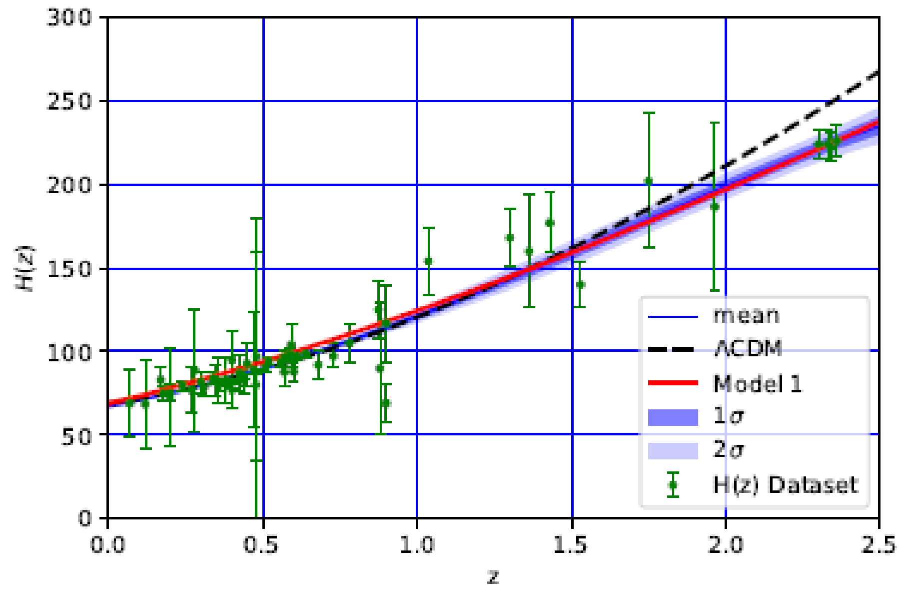
TABLE I. Best fit values of the model parameters

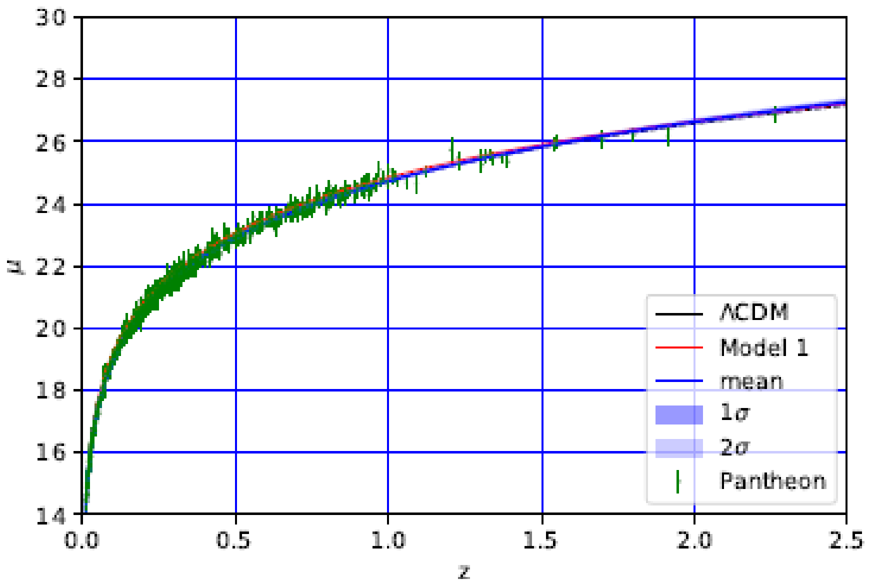


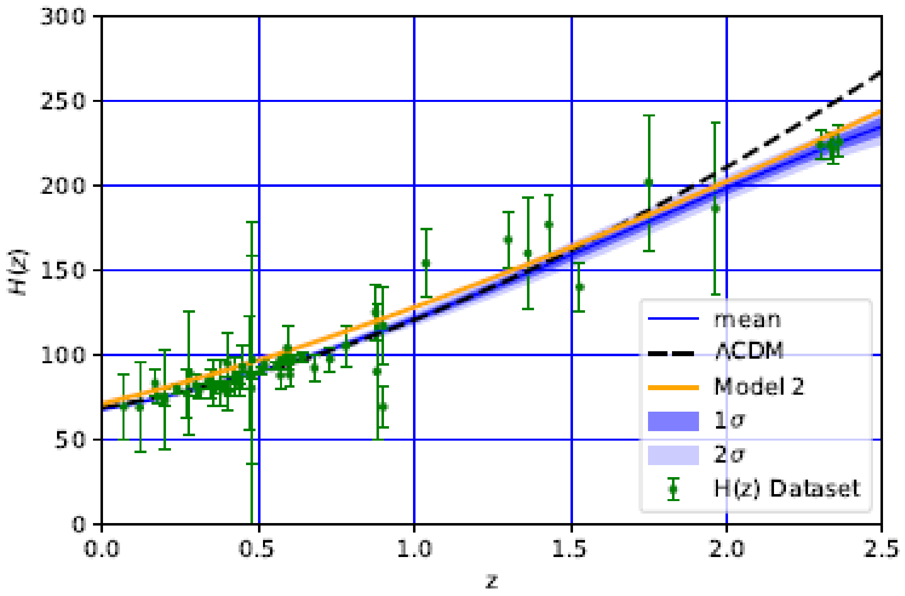


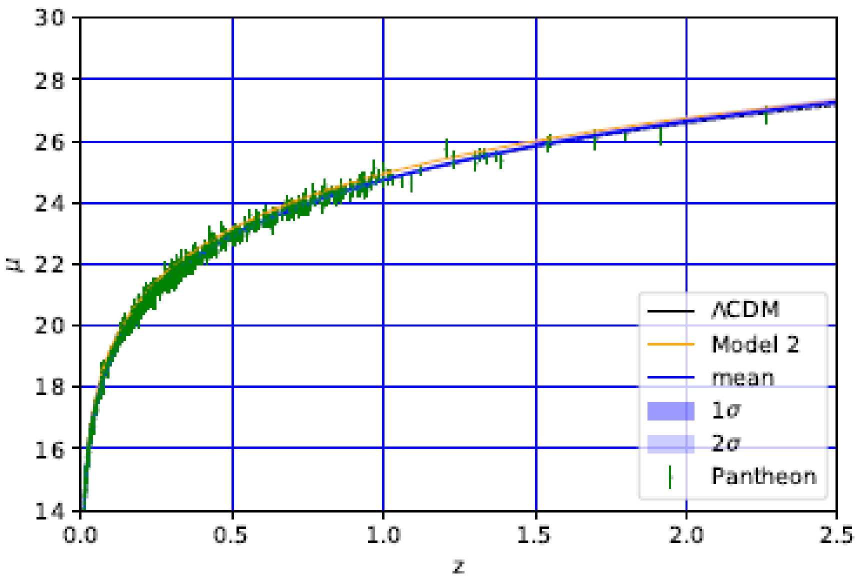
Comparison with the Hubble data points

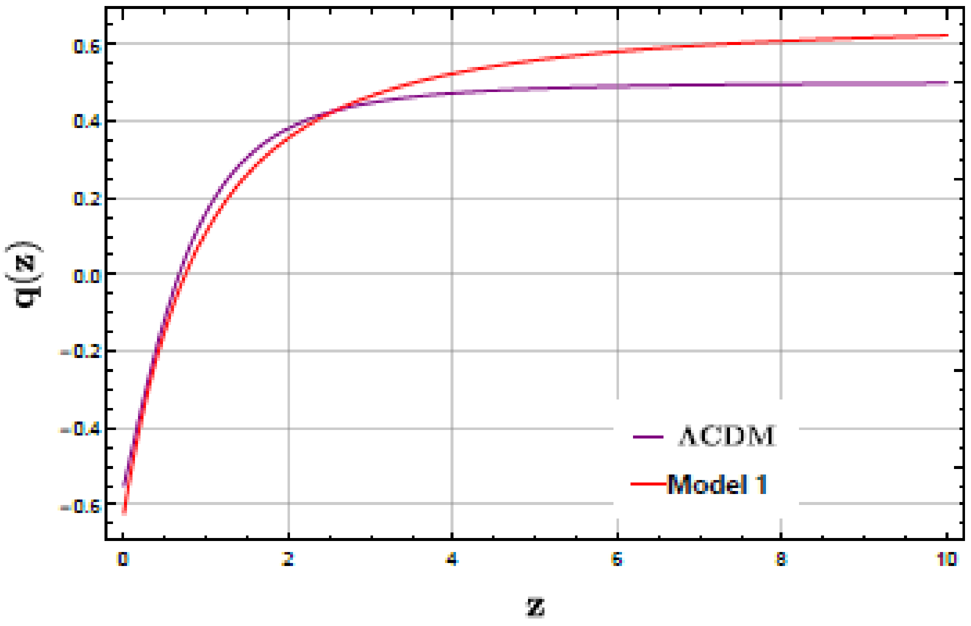
Comparison with the Pantheon data











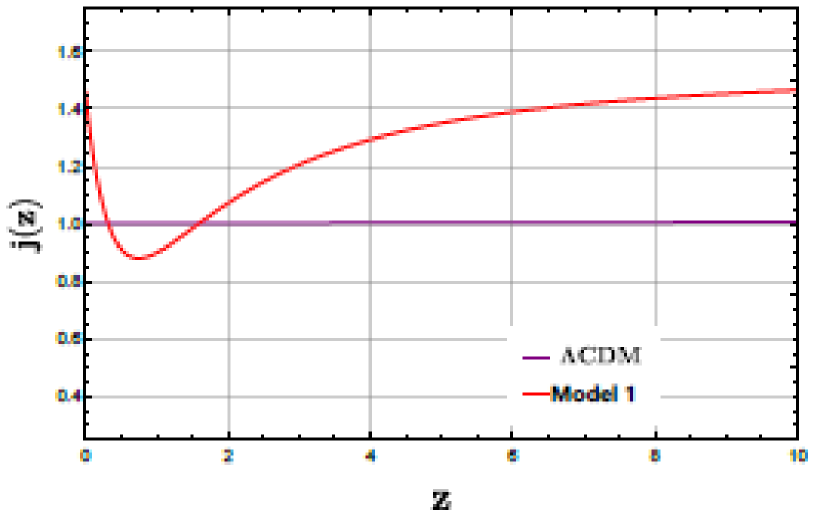
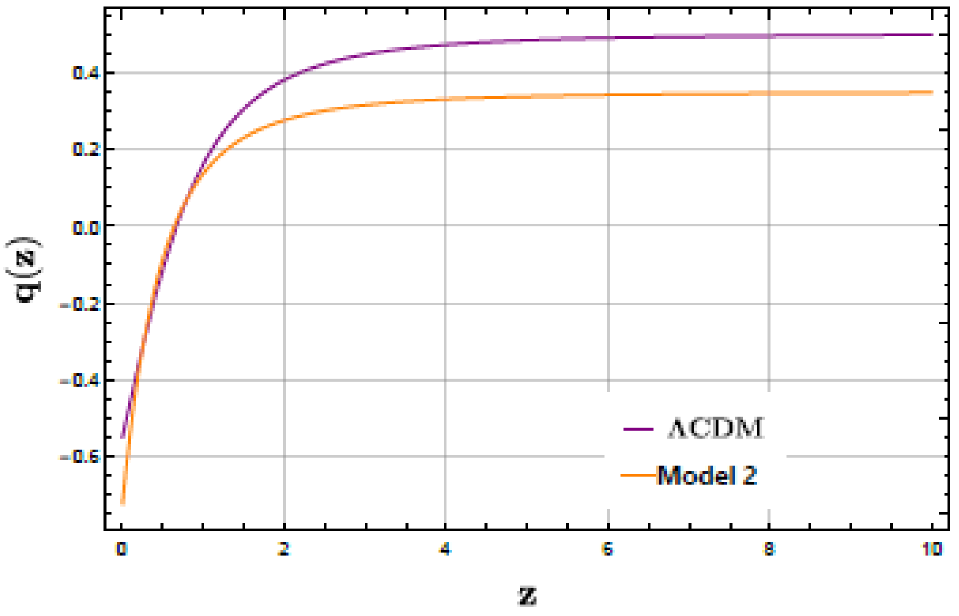


FIG. 11. Evolution of jerk parameter with respect to the redshift of Model 1.



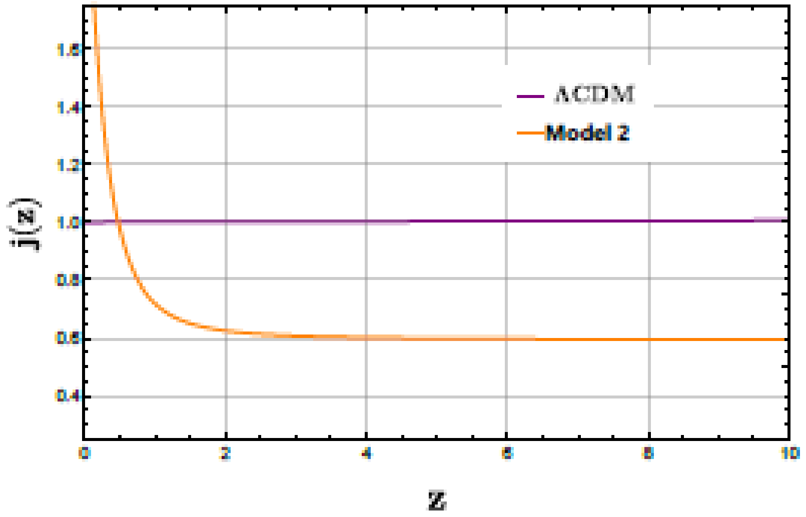


FIG. 12. Evolution of jerk parameter with respect to the redshift of Model 2.

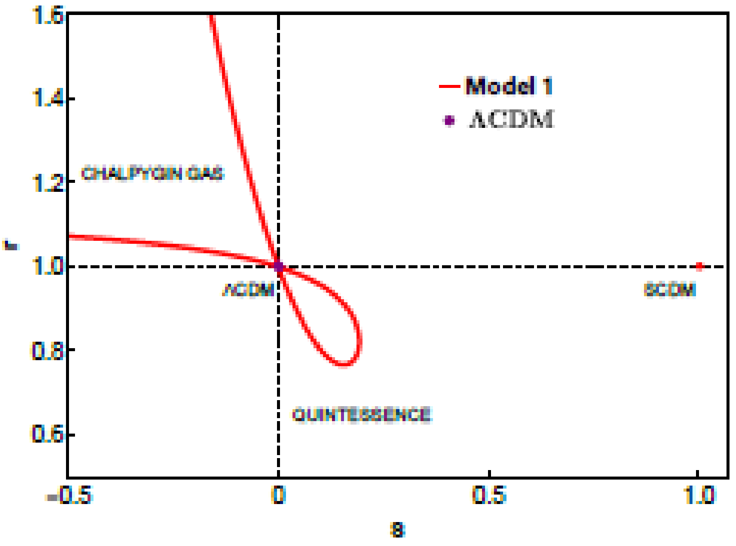


FIG. 15. This figure shows $\{s, r\}$ plots for models 1.

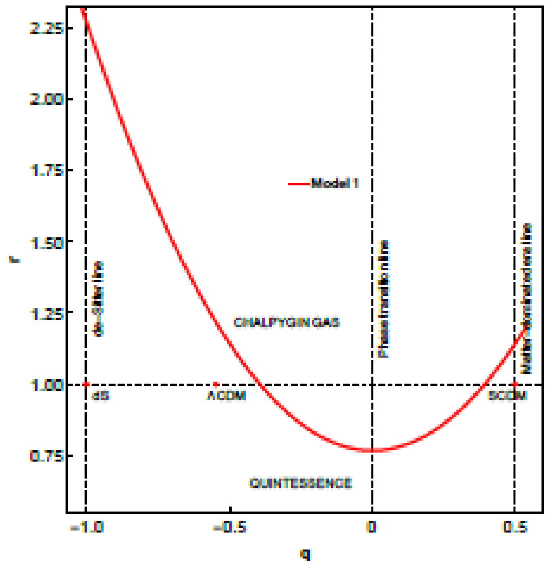


FIG. 16. This figure shows $\{q, r\}$ plots for models 1.

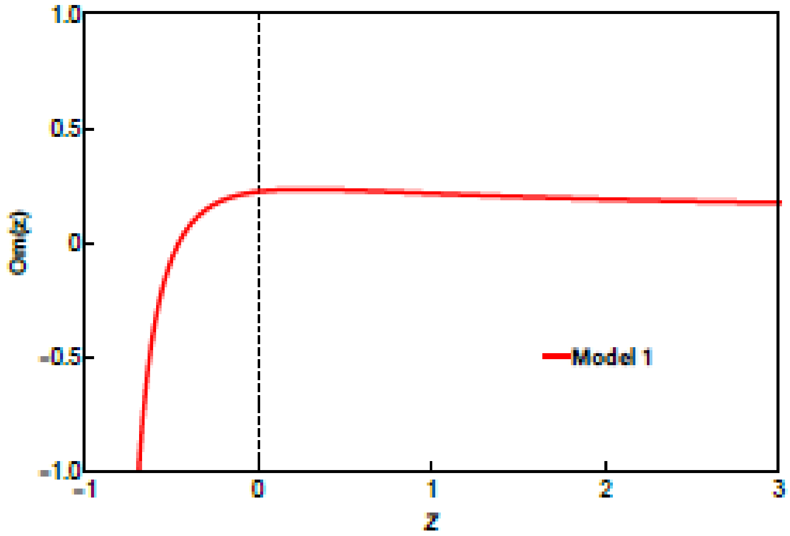


FIG. 19. This figure shows the Om(z) with respect to redshift.

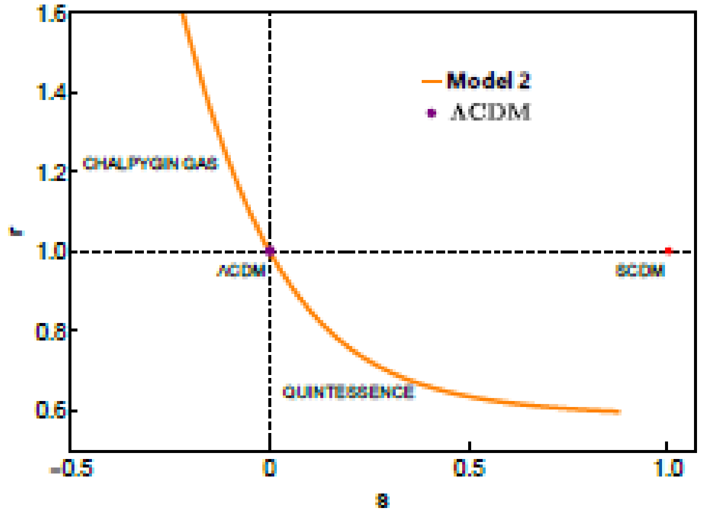


FIG. 17. This figure shows $\{s, r\}$ plots for models 1.

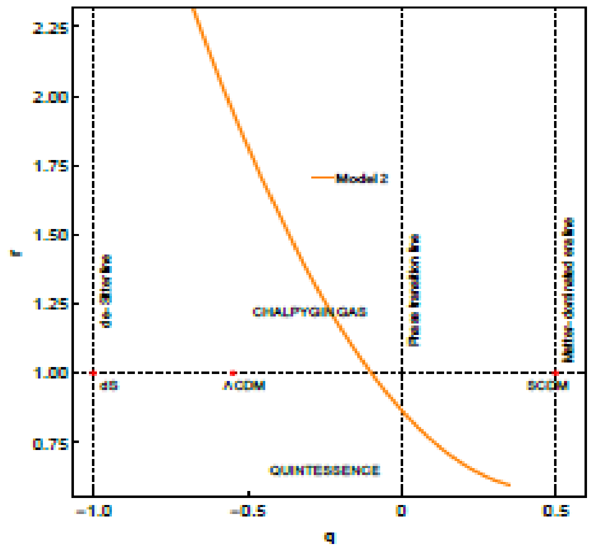


FIG. 18. This figure shows $\{q, r\}$ plots for models 1.

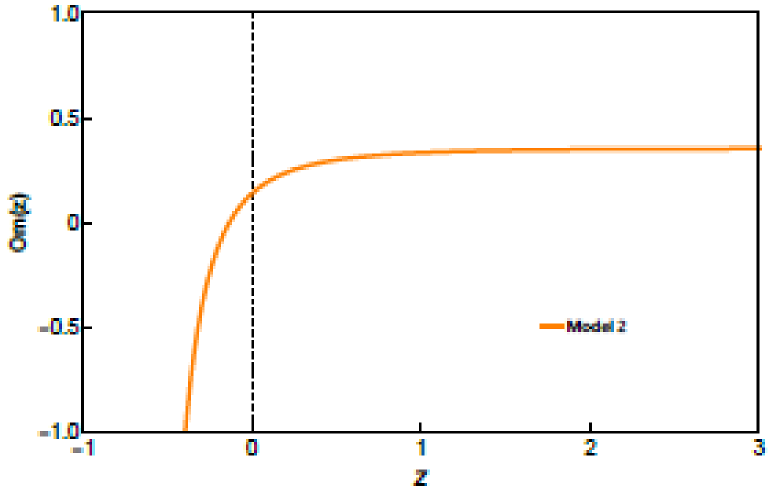


FIG. 20. This figure shows the Om(z) with respect to redshift.

COSMIC EVOLUTION OF PHYSICAL PARAMETERS FOR QUINTESSENCE AS A SOURCE OF DE

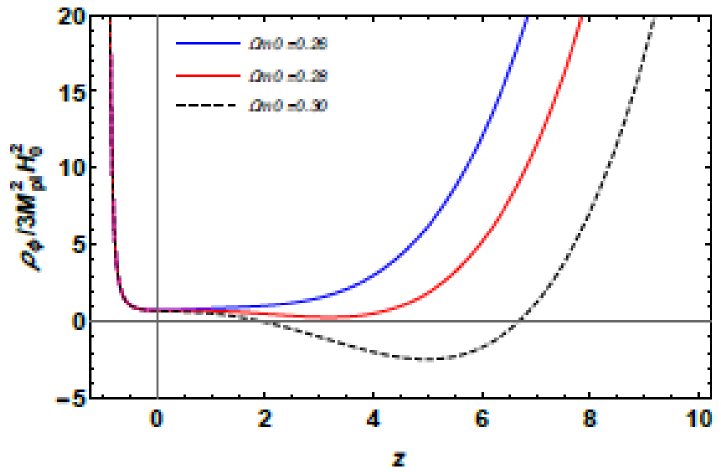


FIG. 21. Profile of energy density for model 1.

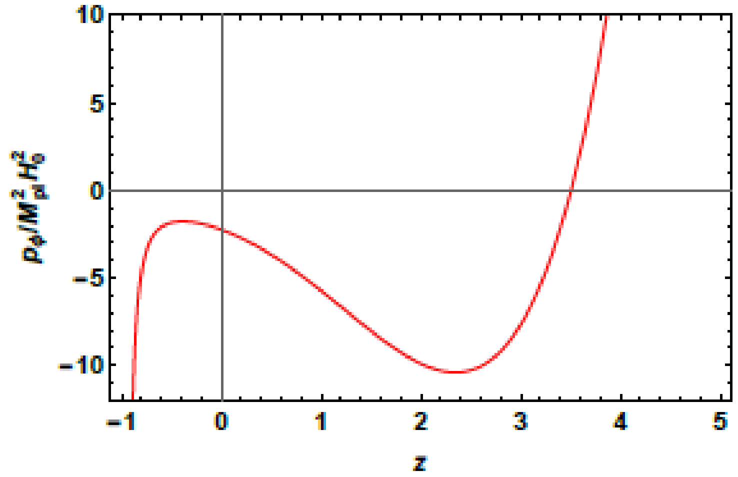


FIG. 22. Profile of dark energy pressure for model 1

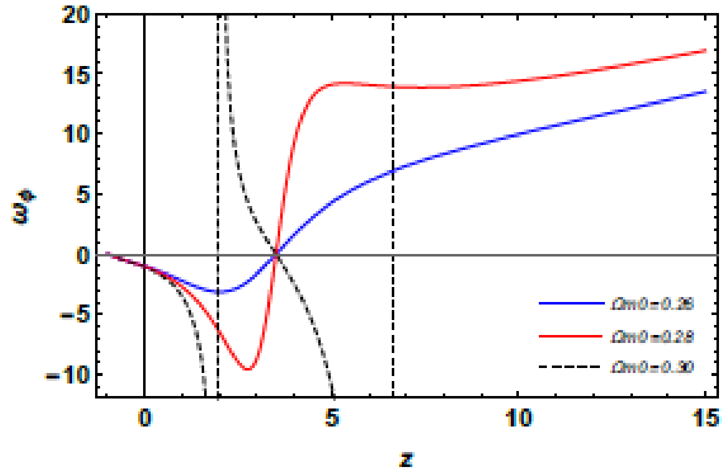


FIG. 25. Profile of dark energy equation of state (EoS) for model 1

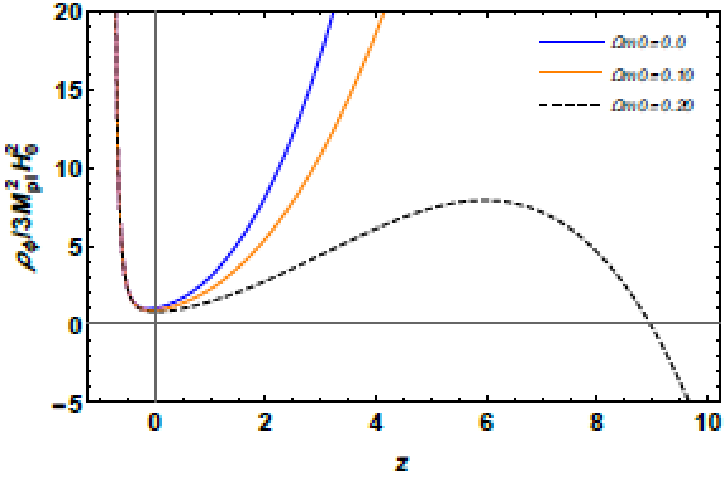


FIG. 23. Profile of energy density for model 2.

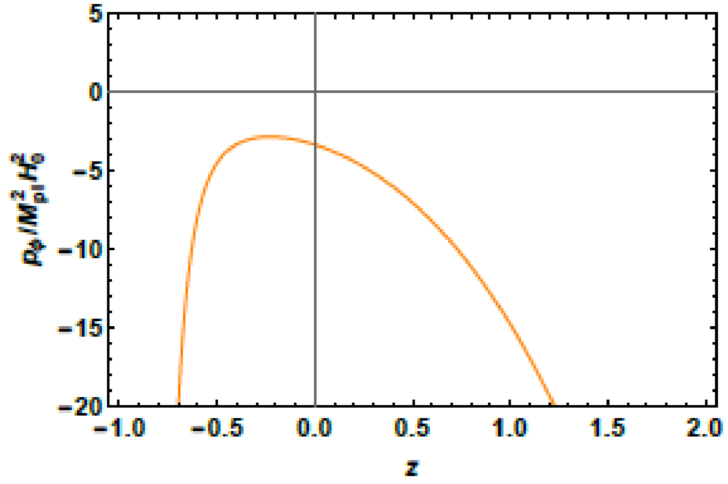


FIG. 24. Profile of dark energy pressure for model 2

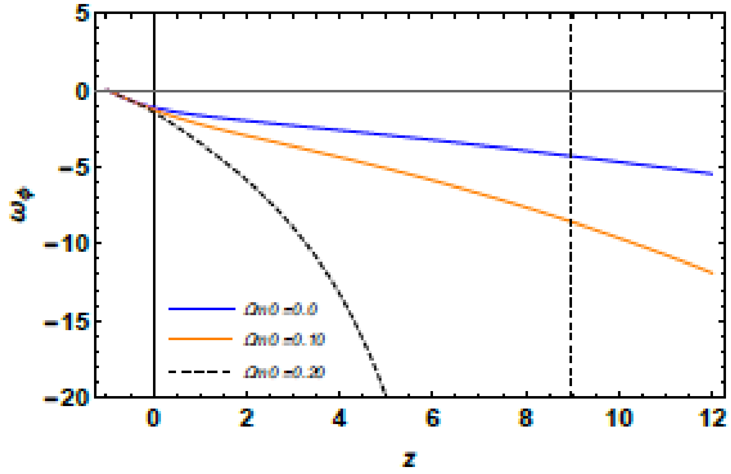
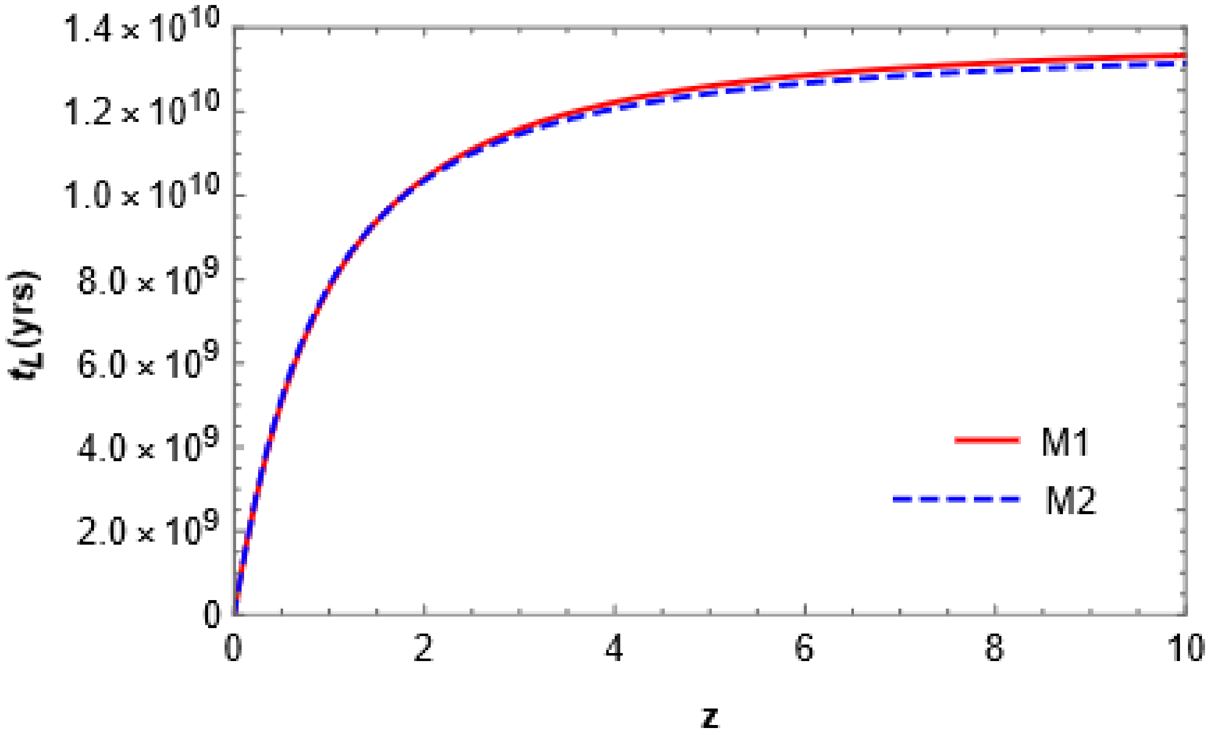


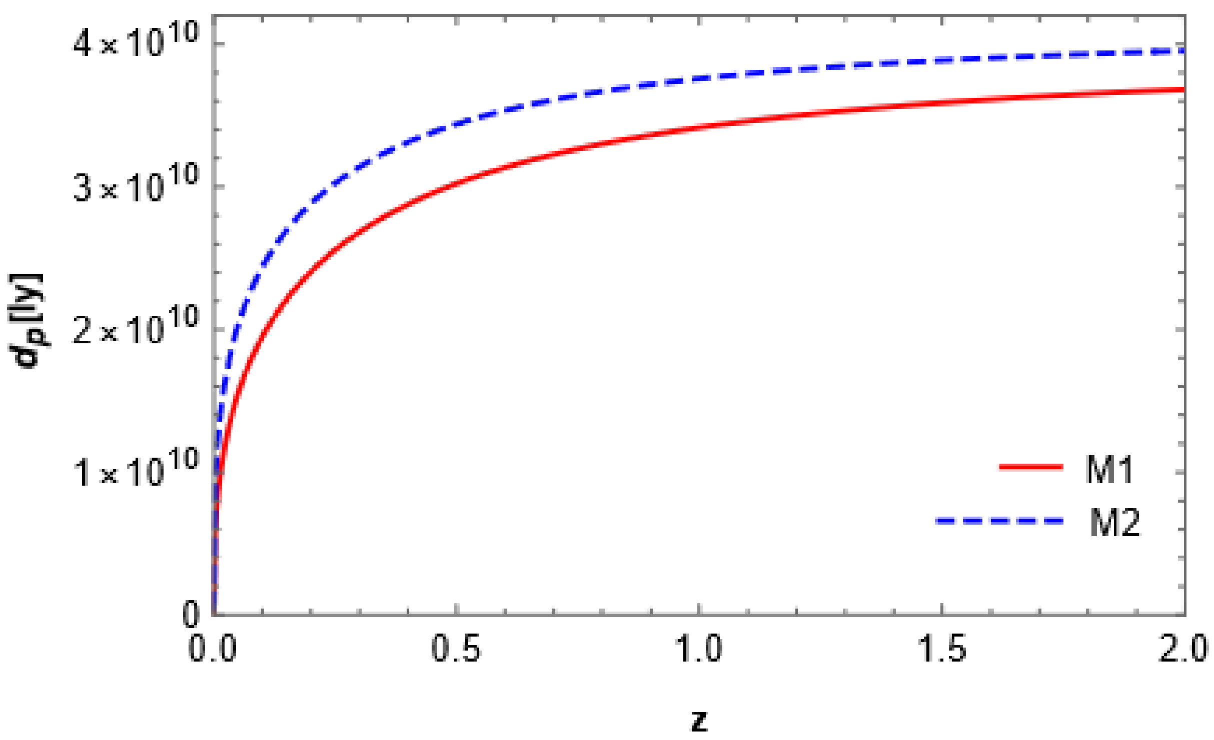
FIG. 26. Profile of dark energy equation of state (EoS) for model 2

KINEMATIC TESTS

A. Lookback Time



B. Proper Distance



C. Luminosity Distance

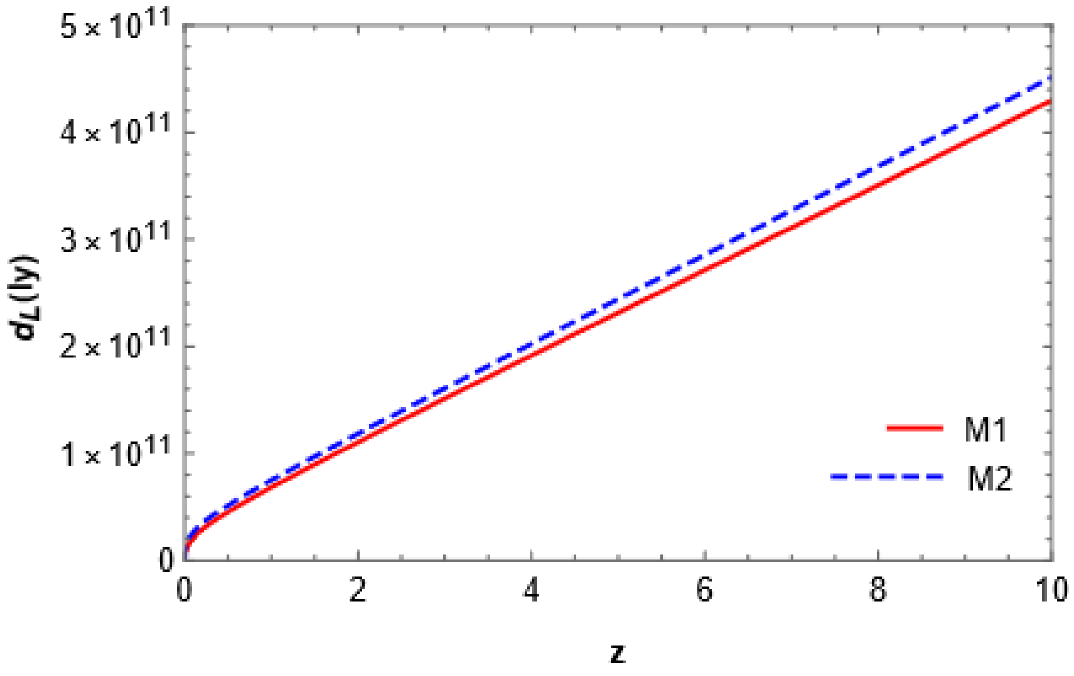
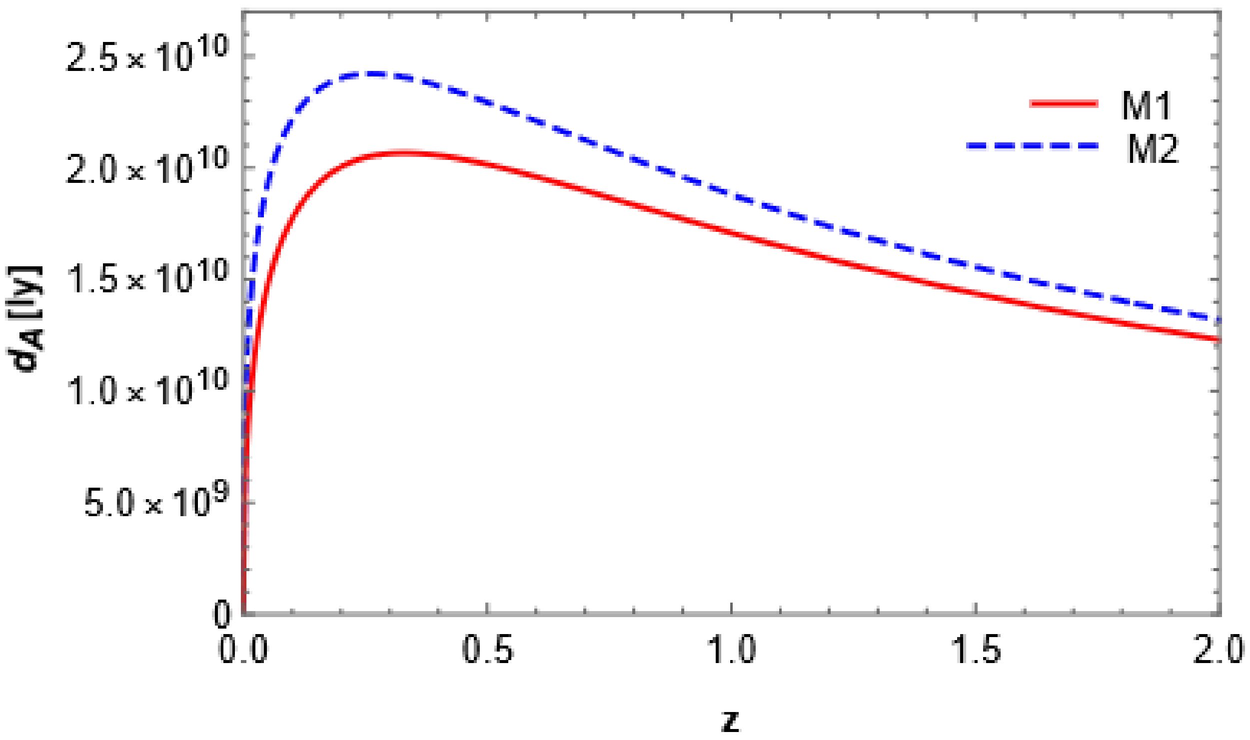


FIG. 23: Luminosity distance in terms of redshift

D. Angular Diameter Distance



MODIFIED GRAVITY

Physical dynamics of dark energy models resulting from a parametrization of H in $f(Q,T)$ gravity

$$S = \int \left[\frac{1}{16\pi} f(Q,T) + \mathcal{L}_M\right] \sqrt{-g}\, d^4x$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}$$

$$\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$$

$$Q \equiv -g^{\mu\nu}(L^{\alpha}_{\beta\mu} L^{\beta}_{\nu\alpha} - L^{\alpha}_{\beta\alpha} L^{\beta}_{\mu\nu})$$

where

$$L^{\alpha}_{\beta\gamma} \equiv -\frac{1}{2} g^{\alpha\lambda}(\nabla_\gamma g_{\beta\lambda} + \nabla_\beta g_{\lambda\gamma} - \nabla_\lambda g_{\beta\gamma})$$

The trace of nonmetricity tensor is,

$$Q_\alpha \equiv Q_\alpha{}^{\mu}{}_{\mu},\ \tilde{Q}_\alpha \equiv Q^{\mu}{}_{\alpha\mu}$$

$$3H^2 = 8\pi\rho_{eff} = \frac{f}{4F} - \frac{4\pi}{F}[(1+\bar{G})\rho + \bar{G}p]$$

$$2\dot{H} + 3H^2 = -8\pi p_{eff} = \frac{f}{4F} - \frac{2\dot{F}H}{F} + \frac{4\pi}{F}[(1+\bar{G})\rho + (2+\bar{G})p]$$

$$f(Q,T) = \mu Q + \nu T$$

for M1

$$\omega = -\Big[\big((1+z)\beta\big)^{\alpha}\Big(8\pi(3-2\alpha)+3\nu(1-\alpha)\Big)+\Big(8\pi(3+2\alpha)+3\nu(1+\alpha)\Big)\Big]/\Big[\Big(24\pi+3\nu-\alpha\nu\Big)+\big((1+z)\beta\big)^{\alpha}\Big(24\pi+3\nu+\alpha\nu\Big)\Big]$$

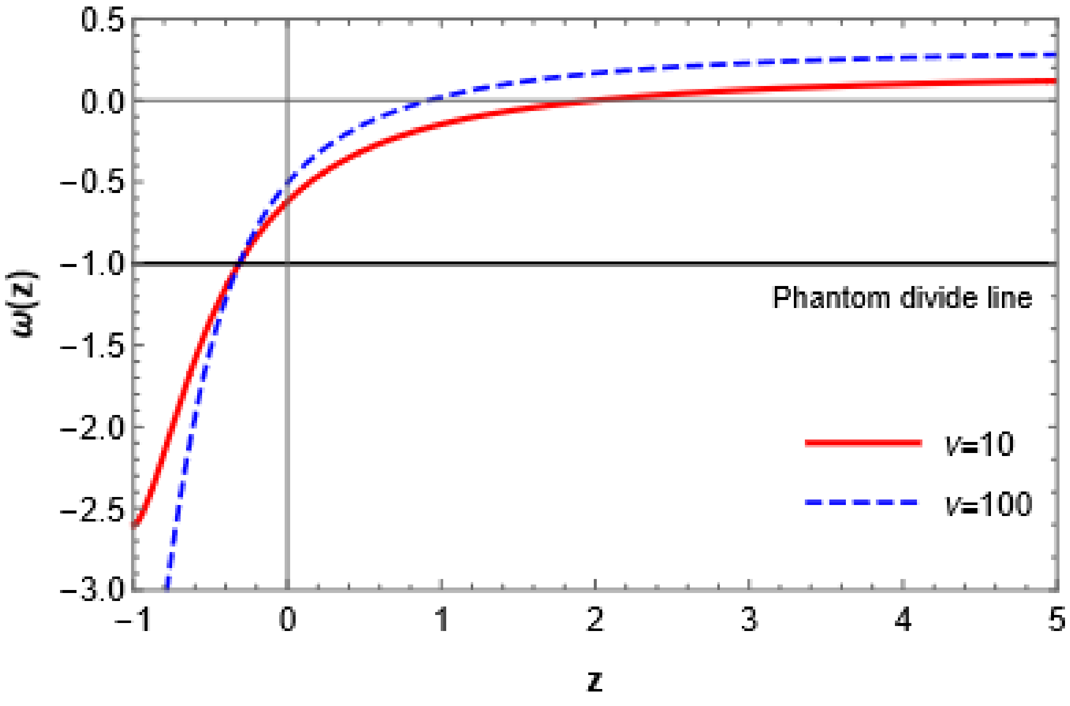


FIG. 2: Equation of state parameter for model M1

for M2

$$\omega = -\Big[\big((1+z)\beta\big)^{2\alpha}\Big(8\pi(3-2\alpha)+3\nu(1-\alpha)\Big)+\Big(8\pi(3+4\alpha)+3\nu(1+\alpha)\Big)\Big]/\Big[\Big(24\pi+3\nu-2\alpha\nu\Big)+\big((1+z)\beta\big)^{2\alpha}\Big(24\pi+3\nu+3\alpha\nu\Big)\Big]$$

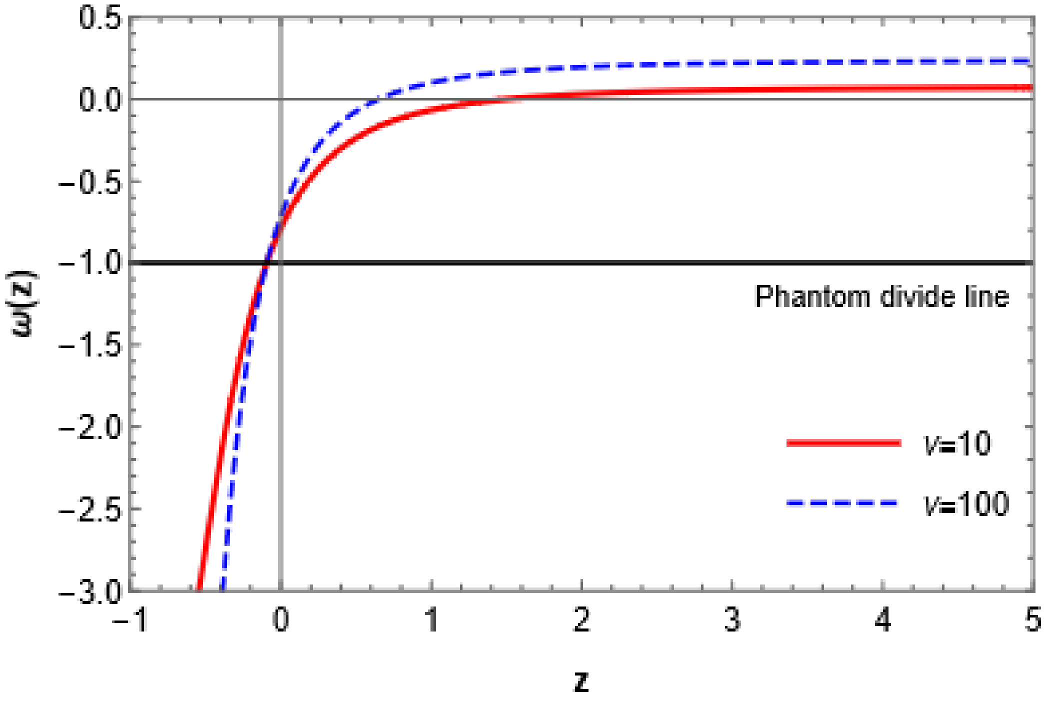


FIG. 3: Equation of state parameter for model M2

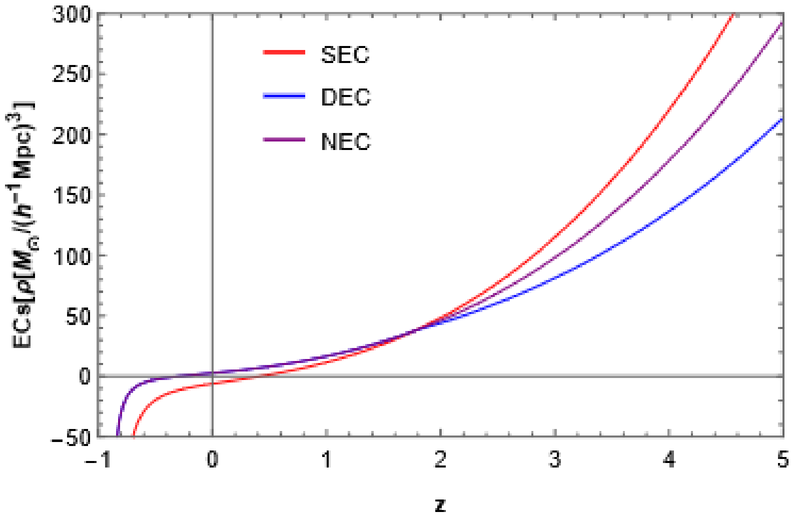


FIG. 5: Energy conditions in terms of redshift for model M1

Geometrical Parameters		
Parameters at $z = 0$	**Model M1**	**Model M2**
Deceleration parameter (q_0)	-0.5372	-0.7462
Jerk parameter (j_0)	1.1996	2.8025
Snap parameter (s_0)	1.4032	8.6445
Lerk parameter (l_0)	38.9871	48.3503

TABLE I: Cosmographic parameters at $z = 0$

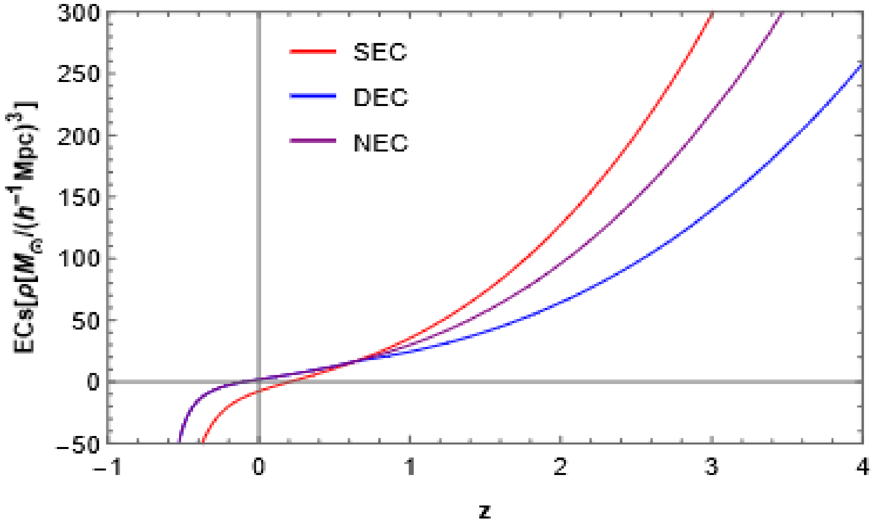


FIG. 12: Energy conditions in terms of redshift for model M2

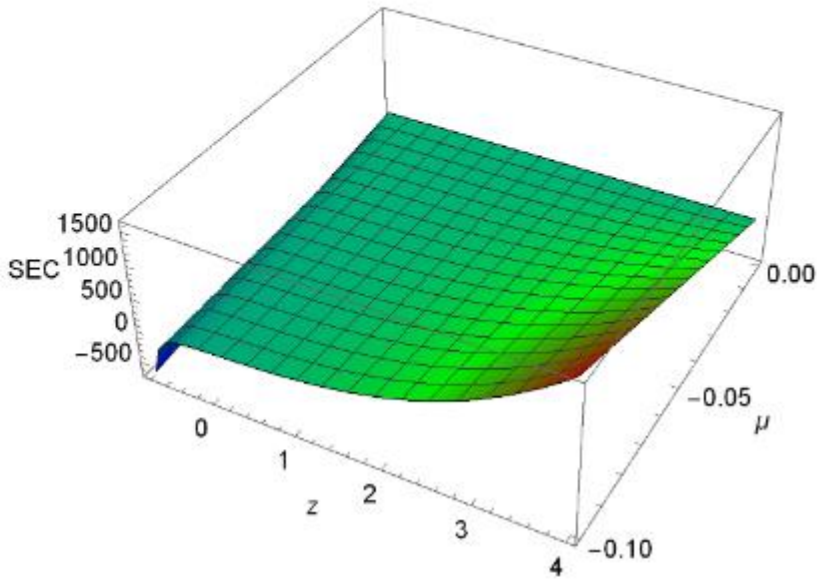


FIG. 6: SEC: variation in μ

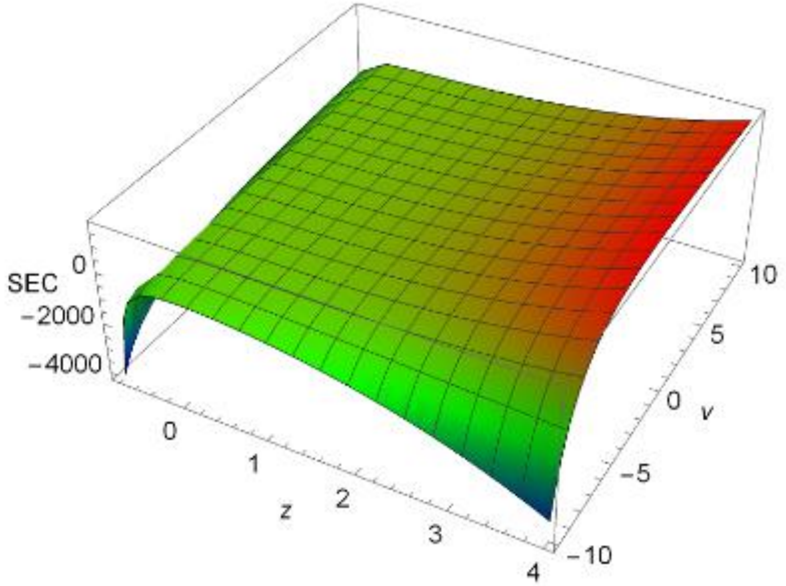


FIG. 7: SEC: variation in ν

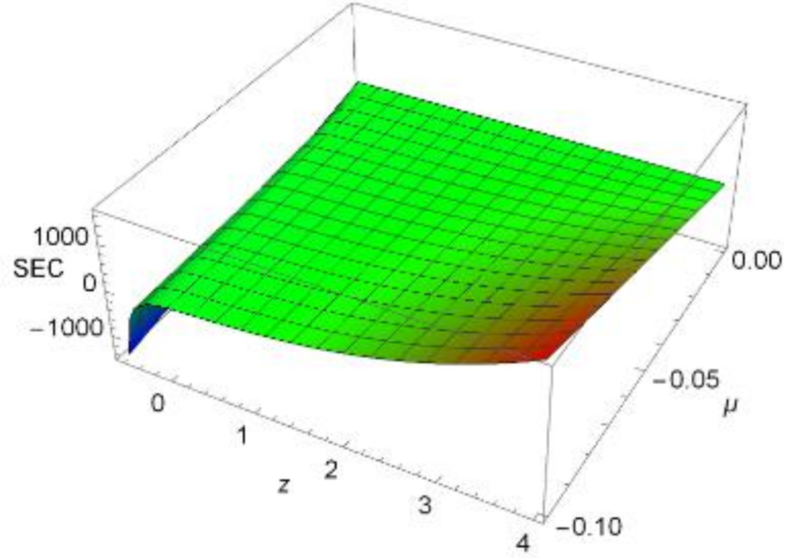


FIG. 13: SEC: variation in μ

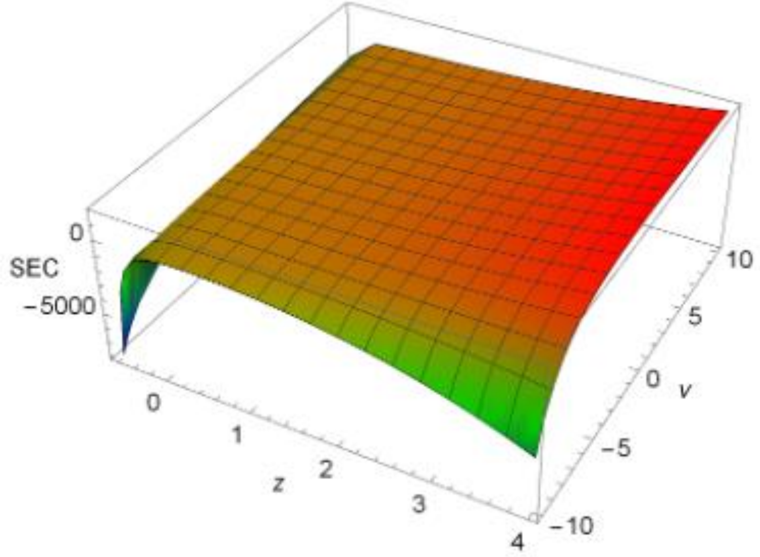


FIG. 14: SEC: variation in ν

CONDITION OF STABILITY

$$c_s^{\,2} = \frac{dp}{d\rho} = \frac{dp/dz}{d\rho/dz}$$

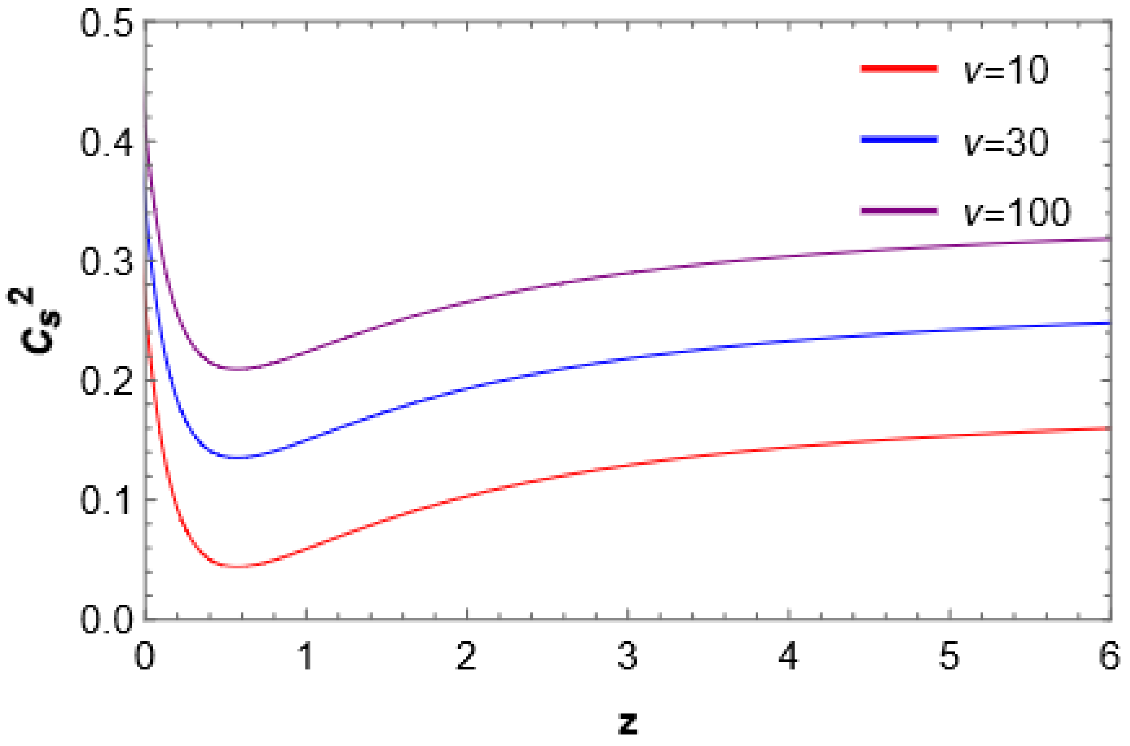


FIG. 25: c_s^2 in terms of redshift for model M1

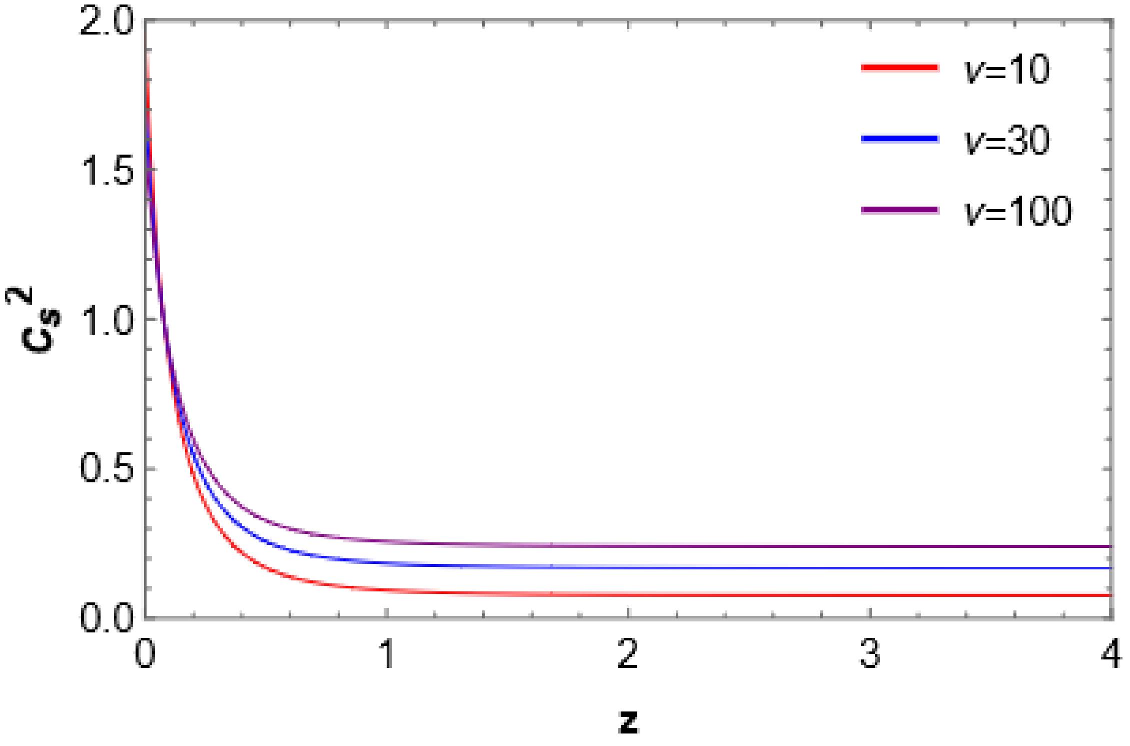


FIG. 26: c_s^2 in terms of redshift for model M2

- **Thus, one can obtain exact solutions to Einstein field equations using the concept of cosmological parametrization (a detailed list have been provided here) for any choice of dark energy to explain the cosmic acceleration as well as to alleviate some standard problems of GR.**
- **The cosmological dynamics can be described for the reconstructed models and the model parameters (and also the integrating constants) are to be constrained through observational datasets.**

THANK YOU