On the quasi-stationary expansion in gravity

Dieter Van den Bleeken
Boğaziçi University

In collaboration with

Based on:

1904.12869
2106.13268
S. Kutluk, A. Seraj

1703.03459
1903.10682
2002.02688
2210.15440
M. Ergen, M. Elbistan, E. Hamamcı, U. Zorba

work in progress
A. Beyen, E. Hamamcı, C. Maes, K. Meerts

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Slow ~~Food~~ Physics

quasi-stationary/adiabatic/Manton/moduli space/... approximation

1) Starting point: $\bar{f}(x;z)$ family of (time-independent) solutions

z: parameters labelling different solutions

- → integration constants
- → external parameters

Slow ~~Food~~ Physics

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2) Ansatz: $f(x, t) = \bar{f}(x; z(t))$

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1) Starting point: $\bar{f}(x;z)$ family of (time-independent) solutions

z: parameters labelling different solutions

- integration constants
- external parameters

2) Ansatz: $f(x,t) = \bar{f}(x;z(t)) + \mathcal{O}(\dot{z})$

3) Insert into equations and solve perturbatively in small $\dot{z}$.

Motion on the space of vacua

- Vacuum: minimal energy equilibrium configuration
  $\Leftrightarrow$ absolute minimum of potential

- Focus on case with manifold of vacua = 'space of vacua' $\mathcal{V}$
  ↳ typical for theories with symmetry breaking: $\mathcal{V} = \mathcal{S}/\mathcal{K}$

- Kinetic Energy defines a metric on configuration space $\mathcal{C}$

  $\mathcal{V} \subset \mathcal{C} \quad \Rightarrow$ induced metric on $\mathcal{V}$

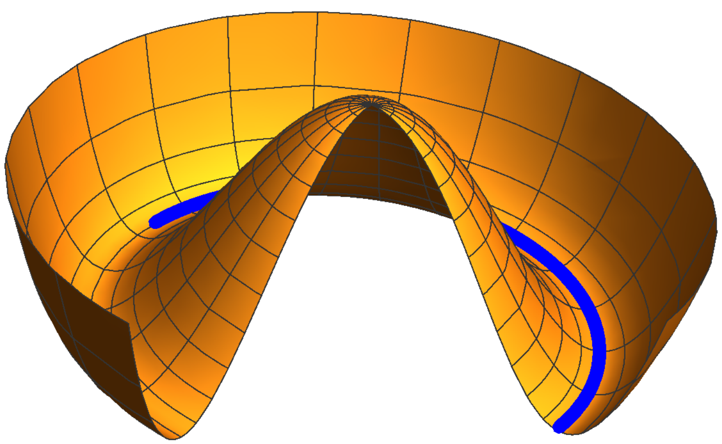
- For small velocities **free motion on** $\mathcal{V}$ $\quad$ $\left[\text{corrections } \mathcal{O}(v^2)\right]$
  ↳ geodesic motion

Motion on the space of vacua

Example I

$$L = \tfrac{1}{2}\mathrm{g}(\dot{x}, \dot{x}) - V(x) = \tfrac{m}{2}\dot{x}^a \dot{x}^a - (R^2 - x^a x^a)^2$$

Mexican Hat potential



$$\mathcal{C} = \mathbb{R}^n$$

$$\mathcal{V} = S^{n-1} = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-1)}$$

centrifugal force $\sim \frac{v^2}{R}$

restoring force $\sim R^2(R - r)$

$$\Rightarrow (R - r) \sim \frac{v^2}{R^3}$$

Motion on the space of vacua

Example II

$$L = \frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2N} - N\frac{(R^2 - \eta_{\mu\nu}x^\mu x^\nu)^2}{2}$$

↑ reparametrization invariant

Gaugefix $N = 1$

$$L = \frac{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2} - \frac{(R^2 - \eta_{\mu\nu}x^\mu x^\nu)^2}{2}$$

$$\mathcal{C} = \mathbb{R}^{n+1}$$

$$\mathcal{V} = \mathrm{dS}_n = \frac{\mathrm{SO}(1,n)}{\mathrm{SO}(1,n-1)}$$

↑ Lorentzian signature

Motion on the space of vacua

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reparametrization invariant

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Lorentzian signature

constraint

$$-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = (\eta_{\mu\nu}x^\nu x^\mu - R^2)^2$$

on $\mathcal{V}$

null geodesic motion

Motion on the space of vacua

GR (on $\mathbb{R} \times M$)

In synchronous gauge/Gaussian normal coordinates

$$\left(ds^2 = dt^2 + h_{ij}(x,t)dx^i dx^j \right)$$

$$L = \frac{1}{2}\mathrm{g}(\dot{h}, \dot{h}) - V(h)$$

$$\mathrm{g}(\delta_1 h, \delta_2 h) = \frac{1}{2}\int_M d^3x \sqrt{h}\, h^{i[k}h^{j]l}\delta_1 h_{ij}\delta_2 h_{kl} \qquad \text{(WdW)}$$

$$V(h) = -\frac{1}{2}\int_M d^3x\,\sqrt{h}\,R(h)$$

$\mathcal{C}$ = space of all Riemannian metrics on M (superspace)

↑ highly singular

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$$L = \frac{1}{2} \mathrm{g}(\dot{h}, \dot{h}) - V(h)$$

+ constraints !
(momentum and Hamiltonian)

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What is $\mathcal{V}$?

Motion on the space of vacua

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What is $\mathcal{V}$?

$$\mathrm{Ric}(h) = 0 \overset{d=3}{\Leftrightarrow} \mathrm{Riem}(h) = 0 \Leftrightarrow h_{ij} = \delta_{ij} \text{ up to diffeo}$$

$$\mathcal{G} = \{\phi \in \mathrm{Diff}(M) \,|\, \phi(\partial M) = \partial M\}$$

gauge symmetry

$$\mathcal{G}_0 = \{\phi \in \mathrm{Diff}(M) \,|\; \phi|_{\partial M} = \mathrm{id}\} \longleftarrow \text{gauge symmetry}$$

$$\mathcal{S} = \mathcal{G}_0 \backslash \mathcal{G} = \mathrm{Diff}(\partial M) \longleftarrow \text{global symmetry}$$

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$$\mathcal{V} = \mathcal{S}/\mathcal{K} = \mathrm{Diff}(\partial M)/\mathrm{ISO}(\partial M) = \mathcal{G}_0 \backslash \mathcal{G}/\mathrm{Stab}(h^o)$$

non-trivial only when $\partial M \neq \emptyset$

perfectly smooth (but ∞-dimensional)

Motion on the space of vacua

Previous slide somewhat axiomatic, physical derivation of $\mathcal{V}$?

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e.g. by small velocity approximation

bonus:

- induced metric on $\mathcal{V}$
- Einstein equations as geodesic motion on $\mathcal{V}$
- infinite number of conserved boundary charges

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setup: $\bar{h}^o_{ij}(x)$ a reference flat metric

ϕ_z a spatial diffeo (z coordinate on $\mathrm{Diff}(M)$)

$\Rightarrow$ generic flat metric $\bar{h}_{ij}(x;z) = \phi_z \cdot \bar{h}^o_{ij}(x)$

Restrict time dependence to $h_{ij}(x,t) = \bar{h}_{ij}(x;z(t))$

determine $z(t)$ in the limit of small $\dot{z}$

Motion on the space of vacua

$$h_{ij}(x,t) = \bar{h}_{ij}(x;z(t)) \quad \Rightarrow \quad \dot{h}_{ij} = \bar{\nabla}_{(i}\chi^z_{j)}$$

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- Momentum constraint: $\nabla^i \partial_{[i}\chi^z_{j]} = 0$

Fixes χ up to boundary value ζ (and an exact part)

$\hookrightarrow \in \mathfrak{diff}(M)/\mathfrak{diff}_0(M) = \mathfrak{diff}(\partial M)$

Motion on the space of vacua

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$$\Rightarrow \quad L = \frac{1}{2}\bar{\mathrm{g}}_z(\dot{z},\dot{z}) \qquad \bar{\mathrm{g}}_z(\dot{z},\dot{z}) = \langle \zeta_z, \zeta_z \rangle_{\bar{h}(z)}$$

$$\langle \zeta_{(1)}, \zeta_{(2)} \rangle_h \equiv \oint_{\partial M} \sqrt{k}\, d^2y \left(\zeta^a_{(1)} D^\perp \zeta^{(2)}_a - K_{ab}\zeta^a_{(1)}\zeta^b_{(2)} \right)$$

$\bar{\mathrm{g}}$ pseudo-Riemannian metric on $\mathrm{Diff}(\partial M)/\mathrm{ISO}(\partial M)$

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$\bar{\mathrm{g}}$ pseudo-Riemannian metric on $\mathrm{Diff}(\partial M)/\mathrm{ISO}(\partial M)$

- Hamiltonian constraint:

  imposes $\dot{z}$ to be null wrt $\bar{\mathrm{g}}$
  determines exact part of χ

- Dynamic Einstein equation $\Leftrightarrow$ geodesic equation for $\bar{\mathrm{g}}$


AND NOW FOR SOMETHING
COMPLETELY DIFFERENT.

The $1/c$ expansion of GR

covariant non-relativistic (small v!) expansion (Dautcourt)

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$$g_{\mu\nu}(c) = \sum_{n=-1}^{\infty} \overset{(2n)}{g}_{\mu\nu}\, c^{-2n} \qquad g^{\mu\nu}(c) = \sum_{n=0}^{\infty} \overset{(2n)}{g}{}^{\mu\nu} c^{-2n}$$

Ansatz: $\overset{(-2)}{g}_{\mu\nu} = -\tau_\mu \tau_\nu$ (natural via $x^0 = ct$)

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- It follows that

$$\overset{(0)}{g}{}^{\mu\nu} = h^{\mu\nu}$$
$$\overset{(0)}{\Gamma}{}^{\lambda}_{\mu\nu} = \overset{(\mathrm{nc})}{\Gamma}{}^{\lambda}_{\mu\nu}$$

(assuming $\overset{(-2)}{\Gamma}{}^{\lambda}_{\mu\nu}=0$)

$(\tau_\mu, h^{\mu\nu})$ with $\tau_\mu h^{\mu\nu} = 0$ forms a Newton-Cartan structure

The $1/c$ expansion of GR

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- and that

Newton-Cartan gravity

$$\overset{(0)}{R}_{\mu\nu} = 8\pi G_{\mathrm{N}} \overset{(0)}{\tilde{T}}_{\mu\nu} \quad \Leftrightarrow \quad \overset{(\mathrm{nc})}{R}_{\mu\nu} = 4\pi G_{\mathrm{N}} \rho\, \tau_\mu \tau_\nu$$

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$\Uparrow$

turns out to be weak gravity assumption

- and that

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relax this assumption

- and that

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covariant non-relativistic (small v!) expansion

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Ansatz: $\overset{\text{(-2)}}{g}_{\mu\nu} = -\tau_\mu \tau_\nu$ (natural via $x^0 = ct$)

- One finds extension of NC $\quad (d\tau = 0 \to \tau \wedge d\tau = 0)$

Torsional connection

$$\overset{\text{(nc)}}{\Gamma}{}^{\lambda}_{\mu\nu} - \overset{\text{(nc)}}{\Gamma}{}^{\lambda}_{\nu\mu} = \mathrm{T}^{\lambda}_{\mu\nu} \neq 0$$

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Physical interpretation

extends Newtonian gravity with time-dilation effects

The $1/c$ expansion of GR

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Bonus: expansion consistent at level of the action

(Hansen, Hartong, Obers)

The $1/c$ expansion of GR

3+1 formulation of the $1/c$ expansion

$$ds^2 = -e^{\psi}(c\,dt + C_i dx^i)^2 + e^{-\psi}\gamma_{ij}dx^i dx^j$$

$g_{\mu\nu} \Leftrightarrow (\psi, C_i, \gamma_{ij})$ Kol-Smolkin decomposition
(dual to ADM decomposition)

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$$L = L_0 + c^{-1}L_1 + c^{-2}L_2$$

$$\begin{aligned}
L_0 &= R + \frac{1}{4}e^{2\psi}C_{ij}C^{ij} - \frac{1}{2}\partial_i\psi\partial^i\psi\,,\\
L_1 &= \Gamma^{ij\,kl}D_iC_j\dot{\gamma}_{kl} + (2\partial^j\psi - e^{2\psi}C^{ij}C_i)\dot{C}_j + C^i\partial_i\psi\dot{\psi}\,,\\
L_2 &= -\Gamma^{ij\,kl}C_i\dot{C}_j\dot{\gamma}_{kl} + e^{-2\psi}\dot{\gamma}\dot{\psi} - 2C^i\dot{C}_i\dot{\psi}\\
&\quad + \frac{1}{4}(e^{-2\psi}\Gamma^{kl\,mn} + \Gamma^{ij\,kl\,mn}C_iC_j)\dot{\gamma}_{kl}\dot{\gamma}_{mn}l\\
&\quad - \frac{1}{2}e^{2\psi}\Gamma^{ij\,kl}C_iC_j\dot{C}_k\dot{C}_- \frac{1}{2}(3e^{-2\psi} + C_iC^i)\dot{\psi}^2\,.
\end{aligned}$$

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Leading order in the $1/c$ expansion:

$$L_{\mathrm{LO}} = L_0 = R + \frac{1}{4}e^{2\psi}C_{ij}C^{ij} - \frac{1}{2}\partial_i\psi\partial^i\psi$$

- no time derivatives
- stationary sector of GR
- $1/c$ expansion is a 'post-stationary' expansion

The $1/c$ expansion of GR

3+1 formulation of the $1/c$ expansion

$$ds^2 = -e^{\psi}(c\,dt + C_i dx^i)^2 + e^{-\psi}\gamma_{ij}dx^i dx^j$$

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Higher order in the $1/c$ expansion:

$$\psi = \sum_{n=0}^{\infty} \overset{(n)}{\psi} c^{-n} \qquad C_i = \sum_{n=0}^{\infty} \overset{(n)}{C}_i c^{-n} \qquad \gamma_{ij} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{ij} c^{-n}$$

$\Rightarrow$ eoms take the form

$$\mathbb{D}^2 \overset{(n)}{\Phi} = \overset{(n)}{\mathbb{S}}[\overset{(n-1)}{\Phi}, \ldots, \overset{(0)}{\Phi}, \partial_t \overset{(n-1)}{\Phi}, \ldots, \partial_t \overset{(0)}{\Phi}, \partial_t^2 \overset{(n-2)}{\Phi}, \ldots, \partial_t^2 \overset{(0)}{\Phi}]$$

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	ψ	C_i	γ_{ij}
c^0	ψ	C_i	γ_{ij}
c^{-1}	χ	B_i	α_{ij}
c^{-2}	ϕ	A_i	β_{ij}
c^{-3}	υ	Z_i	ϖ_{ij}
c^{-4}	τ	Y_i	ϵ_{ij}

$1/c$

Post-stationary

	ψ	C_i	γ_{ij}
c^0	ψ	0	γ_{ij}
c^{-1}	0	B_i	0
c^{-2}	ϕ	0	β_{ij}
c^{-3}	0	Z_i	0
c^{-4}	τ	0	ϵ_{ij}

$1/c^2$

Post-Static

	ψ	C_i	γ_{ij}
c^0	0	0	δ_{ij}
c^{-1}	0	0	0
c^{-2}	$-2U$	0	0
c^{-3}	0	$4U_i$	0
c^{-4}	-2Ψ	0	

Post-Newtonian

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$\Rightarrow$ eoms take the form

$$\mathbb{D}^2 \overset{(n)}{\Phi} = \overset{(n)}{\mathbb{S}}[\overset{(n-1)}{\Phi}, \ldots, \overset{(0)}{\Phi}, \partial_t \overset{(n-1)}{\Phi}, \ldots, \partial_t \overset{(0)}{\Phi}, \partial_t^2 \overset{(n-2)}{\Phi}, \ldots, \partial_t^2 \overset{(0)}{\Phi}]$$

What determines the time dependence?

A dynamic dS black hole

Static SdS metric

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 d\Omega_2^2 \,, \qquad F = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}$$

Solution to

$$L = \sqrt{-g}\left(R - \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi)\right)$$

with

$\Phi = \bar{\Phi}$ ⟵ cst such that $V'(\bar{\Phi}) = 0$

$\Lambda = \frac{V(\bar{\Phi})}{2}$

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with

$\Phi = \bar{\Phi}$ ⟵ cst such that $V'(\bar{\Phi}) = 0$

$\Lambda = \frac{V(\bar{\Phi})}{2}$

(Gregory, Kastor, Traschen)

Parameters: $m, \bar{\Phi}$ —— slow time dependence ⟶ At leading order $m(t), \bar{\Phi}(t)$

$+$ lower order corrections

A dynamic dS black hole

A precise quasi-stationary expansion

- Generic spherically symmetric ansatz

$$ds^2 = -FA^2du^2 + 2HAdudr + \frac{1-H^2}{F}dr^2 + r^2 d\Omega_2^2$$

$$F = F(u,r),\ H = H(u,r),\ A = A(u,r),\ \Phi = \Phi(u)$$

A dynamic dS black hole

A precise quasi-stationary expansion

- Generic spherically symmetric ansatz

$$ds^2 = -FA^2du^2 + 2HAdudr + \frac{1-H^2}{F}dr^2 + r^2d\Omega_2^2$$

- Introduce a small parameter ϵ via slow time $\tau = \epsilon u$

$$F(u,r) = f(\epsilon u, r)\,, \quad H(u,r) = \eta(\epsilon u, r)\,, \quad A(u,r) = a(\epsilon u, r)$$

$$\sqrt{\epsilon}\,\Phi(u) = \varphi(\epsilon u)\,, \quad V(\Phi) = v(\sqrt{\epsilon}\Phi)$$

A dynamic dS black hole

A precise quasi-stationary expansion

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$$\sqrt{\epsilon}\,\Phi(u) = \varphi(\epsilon u)\,, \quad V(\Phi) = v(\sqrt{\epsilon}\Phi)$$

$\Rightarrow$ Einstein equations + scalar eom reduce to

$$\zeta\partial_r a^2 = \epsilon\frac{d}{d\tau}\frac{a^2\zeta^2-1}{f} \qquad \partial_r\left(r(f-1+\frac{v}{6}r^2)\right) = \epsilon\frac{a^2\zeta^2-1}{a^2 f}\frac{r}{-}$$

$$\left(-\eta/a = \zeta = \frac{2\,\partial_\tau f}{r\dot{\varphi}^2}\right)$$

A dynamic dS black hole

A precise quasi-stationary expansion

- Generic spherically symmetric ansatz

$$ds^2 = -FA^2du^2 + 2HAdudr + \frac{1-H^2}{F}dr^2 + r^2 d\Omega_2^2$$

- Introduce a small parameter ϵ via slow time $\tau = \epsilon u$

$$F(u,r) = f(\epsilon u, r)\,, \quad H(u,r) = \eta(\epsilon u, r)\,, \quad A(u,r) = a(\epsilon u, r)$$

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$\Rightarrow$ solve perturbatively: $a = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad f = \sum_{n=0}^{\infty} f_n \epsilon^n$

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$\hookrightarrow$ $\epsilon = 1/c \;\Leftrightarrow\; 1/c$ expansion!

A dynamic dS black hole

At leading order:

$$\partial_r a_0^2 = 0 \qquad \partial_r \left(r(f_0 - 1 + \frac{v}{6} r^2) \right) = 0$$

$$\Rightarrow \qquad a_0 = 1 \qquad f_0 = 1 - \frac{2m(\tau)}{r} - \frac{v(\varphi(\tau))}{6} r^2$$

Observe: Einstein equations leave $m(\tau)$ and $\varphi(\tau)$ undetermined!

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Resolution: (Gregory, Kastor, Traschen)

boundary condition on horizons

$$\varphi \text{ purely ingoing} \qquad \Leftrightarrow \qquad \begin{cases} \dot{m} & = \beta[m, v(\varphi)]\dot{\varphi}^2 \\ \dot{\varphi} & = -\frac{v'(\varphi)}{3\gamma[m, v(\varphi)]} \end{cases}$$

Summary

The quasi-stationary regime of GR

- remains largely unexplored
- is highly tractable
- reveals elegant geometric structures

Interesting open issues

- connection to asymptotic symmetries and soft modes
- how is time dependence determined in $1/c$?
- applications