Modified theories of gravity - a unified approach to metric and metric-affine models

Christian G. Böhmer

c.boehmer@ucl.ac.uk

Department of Mathematics, University College London

GR-QC-Cosmo-Astro online seminar

18 January 2024

Talk is split into 3 main topics

- Foundations of General Relativity and going beyond GR
- Models of modified gravity
- A neat cosmological application

Mathematics of GR – Differential Geometry

The standard formulation of GR is based on various assumptions:

1. 4-dimensional Lorentzian manifold
2. metric structure g
3. connection $\overline{\Gamma}$, or equivalently, a covariant derivative $\overline{\nabla}$
4. derivative is metric compatible $\nabla_\alpha g_{\mu\nu} = 0$
5. derivative is torsion-free $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) f = T_{\alpha\beta}{}^{\gamma} \partial_\gamma f = 0$
6. action linear in curvature

Mathematics of GR – doing more I

Let's dissect these assumptions a bit:

1. **4 dimensions $\longrightarrow$ let's have more!**
   - 1920s Kaluza and Klein, 1 extra spatial dimension, twistors – 4 complex dimensions (=8), bosonic string theory – 26 dimensions, superstring theory – 10 dimensions

3. **Connection: Why $\nabla_\alpha g_{\mu\nu} = 0$? $T_{\alpha\beta}{}^{\gamma} = 0$?**
   - 1920s Einstein-Cartan theory, also Einstein–Cartan–Sciama–Kibble theory 1960s ($\nabla_\alpha g_{\mu\nu} = 0$), late 1920s teleparallel gravity $\mathrm{GR}_{||}$, again ($\nabla_\alpha g_{\mu\nu} = 0$), 1960s Møller, Pellegrini, Plebanski, Hayashi, Nakano; New teleparallel gravity theory or new general relativity ($\nabla_\alpha g_{\mu\nu} = 0$), late 1970s Metric-affine theories ($\nabla_\alpha g_{\mu\nu} \neq 0$, $T_{\alpha\beta}{}^{\gamma} \neq 0$)

Mathematics of GR – doing more II

6. **Action linear in curvature $S_{\mathrm{EH}} = \int R\sqrt{-g}\, d^4x$**

- 1970 Buchdahl 'Non-linear Lagrangians . . . '; 1983 Barrow & Ottewill $f(R)$ cosmology, 1980 Starobinsky, working on inflation; (other than Starobinsky's inflation model these models were largely ignored), late 1990s dark energy, early 2000s $f(R)$ gravity, 2007 $f(T)$ gravity, 2015 $f(T, B)$ gravity [CGB et al], 2017 $f(Q)$ gravity, 2019 $f(G)$ gravity,
- 2021 onward: f of everything theories ☺☺☺

Mathematics of GR – equations of motion

1. in GR the twice contracted Bianchi identities imply the conservation equations $\nabla_\alpha T^{\alpha\beta} = 0$
2. in turn the conservation equations yield (is equivalent to) the equations of motion of matter
3. any theory beyond GR: equations of motion should be derived from its conservation laws
4. we cannot *postulate* geodesics or autoparallels or other curves

Action principle I

All theories discussed henceforth will be based on the action principle.

The Einstein field equations can be derived from the action

$$S_{\text{total}} = \underbrace{\frac{1}{2\kappa^2}\int R\sqrt{-g}\, d^4x}_{S_{\text{EH}}} + \underbrace{\int L_{\text{matter}}(g, \psi, \nabla\psi)\sqrt{-g}\, d^4x}_{S_{\text{matter}}}$$

- R Ricci scalar, g metric determinant, $\kappa^2 = 8\pi G/c^4$ gravitational coupling constant, $\sqrt{-g}\, d^4x$ volume element
- dynamical variable: metric g_{ij}, matter fields ψ

Action principle II - Einstein action

The Einstein field equations can also be derived from the Einstein action

$$S_{\text{total}} = \underbrace{\frac{1}{2\kappa^2}\int g^{\mu\nu}\left(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\lambda\sigma}\right)\sqrt{-g}\, d^4x}_{S_{\text{Einstein}}} + S_{\text{matter}}$$

$$\mathbf{G} = g^{\mu\nu}\left(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\lambda\sigma}\right)$$

- We have the identity $R = \mathbf{G} + \mathbf{B}$ where $\mathbf{B}$ is a boundary term

Action principle III – matter couplings

The moment we write something like $L_{\mathrm{matter}}(g, \psi, \nabla\psi)\sqrt{-g}\, d^4x$ we assume the *minimal* gravitational coupling procedure. This means

Minkowski	$\eta_{ij} \longrightarrow g_{ij}$	general metric
partial	$\partial_i \longrightarrow \nabla_i$	covariant, contains the connection

Non-minimal couplings contain terms which directly connect matter fields to geometrical quantities, examples are

Brans-Dicke	ψR
Bertolami, CGB, Lobo & Harko	$f_2(R) L_{\mathrm{matter}}$
Harko et al + various authors	$f(R,T) \quad T = g^{ij} T_{ij}^{(\mathrm{matter})}$
Gubitosi & Linder	$G^{ij}\nabla_i\psi\nabla_j\psi$ and many more

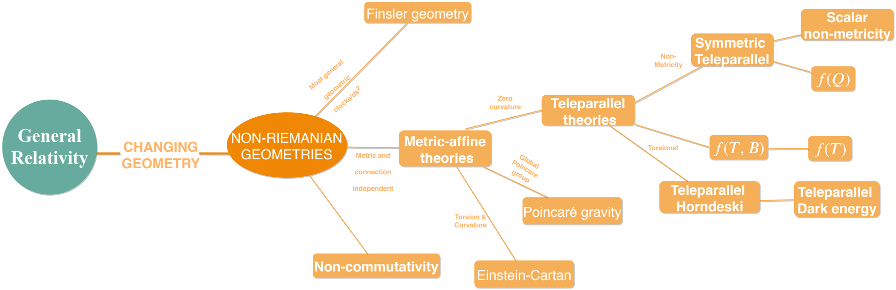
Action principle IV – matter couplings

Non-minimal couplings vanish in the limit of special relativity, no curvature etc. Very hard to constrain using observations.

Couplings to *boundary and topological terms* are interesting. If $\mathcal{G}$ is the Gauss-Bonnet term one can consider $f(\psi)\mathcal{G}$. When $\mathcal{G}$ is added to the Einstein-Hilbert action it does not contribute to the field equations, $\mathcal{G}$ is a topological term.

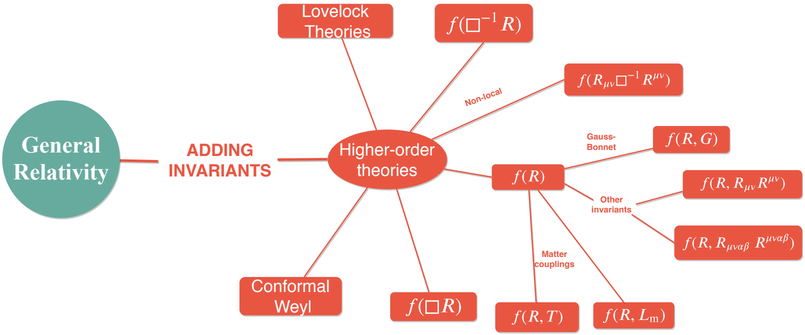
Likewise, if $\mathcal{B}$ is a boundary term (not necessarily related to any topological information) then adding $\mathcal{B}$ to the Einstein-Hilbert action also does not contribute to the field equations, one can again study terms like $f(\psi)\mathcal{B}$.

Modified gravity models I



Source: https://arxiv.org/abs/2105.12582

Modified gravity models II



Source: `https://arxiv.org/abs/2105.12582`

Modified gravity models III – our approach

The starting point of these theories is again the Einstein-Hilbert-like action

$$S_{\mathrm{metric-affine}} = \frac{1}{2\kappa^2} \int \overline{R}\sqrt{-g}\, d^4x$$

The metric-affine Ricci scalar $\overline{R} = \overline{\mathbf{G}} + \overline{\mathbf{B}}$ can be written in two (there are others) different ways which are useful for our approach:

$$\overline{\mathbf{G}} := g^{\mu\lambda}\left(\overline{\Gamma}^{\kappa}_{\kappa\rho}\overline{\Gamma}^{\rho}_{\mu\lambda} - \overline{\Gamma}^{\kappa}_{\mu\rho}\overline{\Gamma}^{\rho}_{\kappa\lambda}\right) + \left(\overline{\Gamma}^{\mu}_{\mu\lambda}\delta^{\nu}_{\kappa} - \overline{\Gamma}^{\nu}_{\kappa\lambda}\right)\left(\partial_{\nu} g^{\kappa\lambda} - \frac{1}{2} g_{\alpha\beta} g^{\kappa\lambda}\partial_{\nu} g^{\alpha\beta}\right)$$

$$\overline{\mathbf{B}} := \frac{1}{\sqrt{-g}}\partial_{\kappa}\left(\sqrt{-g}(g^{\mu\lambda}\overline{\Gamma}^{\kappa}_{\mu\lambda} - g^{\kappa\lambda}\overline{\Gamma}^{\mu}_{\mu\lambda})\right)$$

Modified gravity models IV – our approach

or similarly

$$\overline{R} = \overbrace{\mathbf{G} + \mathbf{B}}^{R} + T - B_T + Q + B_Q + \mathbf{C}$$

where

- T is the torsion scalar $\longrightarrow$ see $f(T)$ gravity
- Q is the non-metricity scalar $\longrightarrow$ see $f(Q)$ gravity
- B_T and B_Q are their respective boundary terms
- $\mathbf{C}$ is a torsion-non-metricity cross term

This decomposition allows us to link various modified theories of gravity.

UCL

Modified gravity models V – independent variables

One now has to make an important decision: Choose independent variable.

- GR one chooses the metric g or tetrad e
- spinorial matter $\Rightarrow$ the matter action contains the connection $\Rightarrow$ we have two possibilities
  - treat $\Gamma = \Gamma(g(e), e)$ and vary with respect to e, standard GR
  - treat g, Γ as *independent*, vary with respect to g and Γ separately; this gives a different theory, Einstein-Cartan theory
- metric-affine/Palatini approach means: treat g and Γ as independent
- (also) metric affine: decompose connection Γ into Levi-Civita part, torsion part, non-metricity part then treat $g, \Gamma_{\rm LC}(g), T, Q$ as independent

Modified gravity models VI

We mentioned previously: early 2000s $f(R)$ gravity, 2007 $f(T)$ gravity, 2015 $f(T, B)$ gravity [CGB et al], 2017 $f(Q)$ gravity, 2019 $f(G)$ gravity ...

The key result is: These theories are not all independent, somewhat contrary to how this is often presented.

First I will discuss the relation between $f(R)$ and $f(T)$; then we will include $f(Q)$. Look at those similar field equations!

$$f'(\mathbf{G})\left[G_{\rho\sigma} + \frac{1}{2}g_{\rho\sigma}\mathbf{G}\right] + \frac{1}{2}f''(\mathbf{G})E_{\rho\sigma}{}^{\gamma}\partial_{\gamma}\mathbf{G} - \frac{1}{2}g_{\rho\sigma}f(\mathbf{G}) = \kappa T_{\rho\sigma}\,,$$

$$f'(T)\left[G_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}T\right] + f''(T)\mathscr{S}_{\rho\sigma}{}^{\gamma}\partial_{\gamma}T + \frac{1}{2}g_{\rho\sigma}f(T) = \kappa\Theta_{\rho\sigma}\,,$$

$$f'(Q)\left[G_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}Q\right] + 2f''(Q)P^{\lambda}{}_{\rho\sigma}\partial_{\lambda}Q + \frac{1}{2}g_{\rho\sigma}f(Q) = \kappa T_{\rho\sigma}\,,$$

Palatini approach

Complete modified action with arbitrary function of both the first order term and the boundary term

$$S_{\mathrm{mod}}[g,\overline{\Gamma}] = \frac{1}{2\kappa}\int f(\overline{\mathbf{G}},\overline{\mathbf{B}})\sqrt{-g}\,d^4x + S_{\mathrm{matter}}$$

with the variations δg and $\delta\Gamma$ leading to

$$-\frac{1}{2}g_{\mu\nu}f + f_{,\overline{\mathbf{G}}}\left(\overline{G}_{(\mu\nu)} + \frac{1}{2}g_{\mu\nu}\overline{\mathbf{G}}\right) + \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\lambda}\partial_{\lambda}f_{,\overline{\mathbf{G}}} + \frac{1}{2}g_{\mu\nu}f_{,\overline{\mathbf{B}}}\overline{\mathbf{B}} - \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\lambda}\partial_{\lambda}f_{,\overline{\mathbf{B}}} = \kappa T_{\mu\nu}$$

$$P^{\mu\nu}{}_{\lambda}f_{,\overline{\mathbf{G}}} + 2\partial_{\rho}f_{,\overline{\mathbf{B}}}\delta_{\lambda}^{[\mu}g^{\rho]\nu} = 2\kappa\Delta^{\mu\nu}{}_{\lambda}$$

Einstein-Cartan model I

Formulate a modified Einstein-Cartan type model, assuming $f = f(\overline{\mathbf{G}})$ so that $f_{,\overline{\mathbf{B}}} = 0$. This gives

$$-\frac{1}{2}g_{\mu\nu}f + f_{,\overline{\mathbf{G}}}\big(\overline{G}_{(\mu\nu)} + \frac{1}{2}g_{\mu\nu}\overline{\mathbf{G}}\big) + \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\lambda}\partial_\lambda f_{,\overline{\mathbf{G}}} = \kappa T_{\mu\nu}$$

$$P^{\mu\nu}{}_{\lambda} f_{,\overline{\mathbf{G}}} = 2\kappa\Delta^{\mu\nu}{}_{\lambda}$$

Next, we wish to set non-metricity to zero. However, one cannot simply set $Q = 0$ in the above!

Einstein-Cartan model II

To do this correctly we need a Lagrange multiplier in the action

$$S_Q = \int \frac{1}{2}\lambda^{\mu\nu\rho}\nabla_\mu g_{\nu\rho}\sqrt{-g}\, d^4x$$

Variations with respect to λ ensure $\nabla_\mu g_{\nu\rho} = 0$. The new field equations thus become

$$-\frac{1}{2}g_{\mu\nu}f + f_{,\overline{\mathbf{G}}}\left(\overline{G}_{(\mu\nu)} + \frac{1}{2}g_{\mu\nu}\overline{\mathbf{G}}\right) + \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\rho}\partial_\rho f_{,\overline{\mathbf{G}}}$$
$$= \kappa\left({}^{(\overline{\Gamma})}T_{\mu\nu} + (\nabla_\rho + T^{\sigma}{}_{\rho\sigma})\Delta^{\rho}{}_{(\mu\nu)}\right)$$
$$P_{\mu\nu\rho}f_{,\overline{\mathbf{G}}} = 2\kappa\Delta_{\mu[\nu\rho]}\,.$$

Einstein-Cartan model with boundary term

We now assume $f_{,\overline{\mathbf{B}}} \neq 0$. Then the field equations are

$$-\frac{1}{2}g_{\mu\nu}f + f_{,\overline{\mathbf{G}}}\left(\overline{G}_{(\mu\nu)} + \frac{1}{2}g_{\mu\nu}\overline{\mathbf{G}}\right) + \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\lambda}\partial_\lambda f_{,\overline{\mathbf{G}}} + \frac{1}{2}g_{\mu\nu}f_{,\overline{\mathbf{B}}}\overline{\mathbf{B}} - \frac{1}{2}\overline{E}_{(\mu\nu)}{}^{\lambda}\partial_\lambda f_{,\overline{\mathbf{B}}} = \kappa\left({}^{(\overline{\Gamma})}T_{\mu\nu} + (\nabla_\lambda + T^{\sigma}{}_{\lambda\sigma})\Delta^{\lambda}{}_{(\mu\nu)}\right)$$

and the other equation is

$$P_{\mu\nu\rho}f_{,\overline{\mathbf{G}}} + 2g_{\mu[\rho}\partial_{\nu]}f_{,\overline{\mathbf{B}}} = 2\kappa\Delta_{\mu[\nu\rho]}$$

Boundary terms ⇒ torsion does not need sources

Let us look at the connection field equation with $\Delta_{\mu[\nu\rho]} = 0$

$$P_{\mu\nu\rho} f_{,\overline{\mathbf{G}}} + 2 g_{\mu[\rho} \partial_{\nu]} f_{,\overline{\mathbf{B}}} = 0$$

This equation is algebraic in the Palatini tensor, this tensor can be written in terms of torsion so that

$$T^{\mu}{}_{\lambda\nu} = \frac{1}{f_{,\overline{\mathbf{G}}}} \delta^{\mu}_{[\lambda} \partial_{\nu]} f_{,\overline{\mathbf{B}}} \, .$$

Therefore, torsion does not vanish in general in source-free regions of spacetime.

Neat cosmological application I

Standard FLRW metric $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$

In cosmology torsion has only 2 independent components
$T^1{}_{10} = T^2{}_{20} = T^3{}_{30} =: h(t)/a^3(t)$ and $T^1{}_{23} = T^2{}_{31} = T^3{}_{12} =: k(t)$.

We consider the simple model $f(\overline{\mathbf{G}}, \overline{\mathbf{B}}) = \overline{\mathbf{G}} + F(\overline{\mathbf{B}})$ and find the following three field equations:

Neat cosmological application II

$$\frac{1}{2}F + 3\frac{\dot{\mathfrak{h}}}{a^3}F' + 3\frac{\mathfrak{h}^2}{a^6} = \rho$$

$$-\frac{1}{2}F - 3\frac{\dot{\mathfrak{h}}}{a^3}F' + \frac{12\mathfrak{h}}{a^6}\left(\ddot{\mathfrak{h}} - 3\frac{\dot{a}}{a}\dot{\mathfrak{h}}\right)F'' + 2\frac{\dot{\mathfrak{h}}}{a^3} - \frac{\mathfrak{h}^2}{a^6} + 4\frac{\mathfrak{h}}{a^3}\frac{\dot{a}}{a} = p = w\rho$$

where $h(t) = \dot{a}a^2 + \mathfrak{h}(t)$ and the connection equation

$$6\left(\frac{\ddot{\mathfrak{h}}}{a^3} - 3\frac{\dot{a}}{a}\frac{\dot{\mathfrak{h}}}{a^3}\right)F'' - 2\frac{\mathfrak{h}}{a^3}2 - \frac{\dot{a}}{a} = 0$$

Nicely, one can eliminate F from the equations

$$\dot{\mathfrak{h}} = \frac{1}{2}(1+w)a^3\rho - 3\frac{\mathfrak{h}^2}{a^3}.$$

Neat cosmological application III

Finally we choose $F(\overline{\mathbf{B}}) = -\beta\overline{\mathbf{B}}^2$ and $w = 0$. Everything pretty much boils down to understanding

$$\frac{\dot{a}}{a} = \frac{\sqrt{2}}{2\sqrt{3\beta}}\frac{Y\sqrt{Y^2+2Y-3}}{Y^2+4Y-1} \qquad Y^2 = 1 + \frac{36\beta\rho_0}{a^3}$$

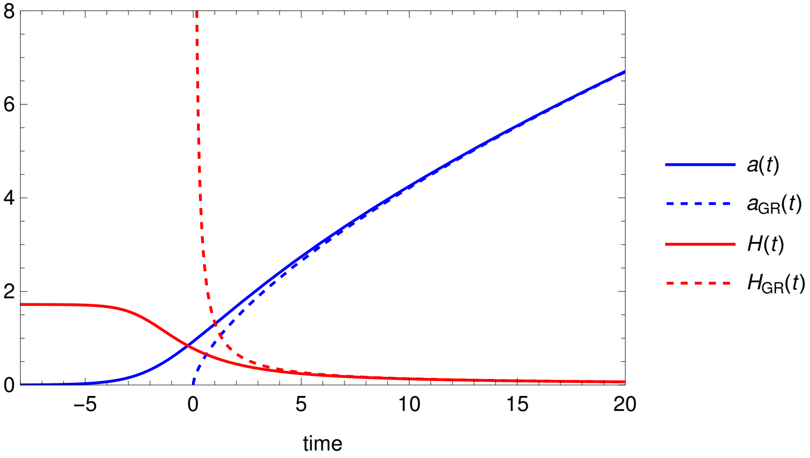
Assuming $a(t) \gg 1$ (late-time universe)

$$\frac{\dot{a}}{a} = \frac{\sqrt{\rho_0}}{\sqrt{3}}\frac{1}{a^{3/2}} + O(a^{-5/2})$$

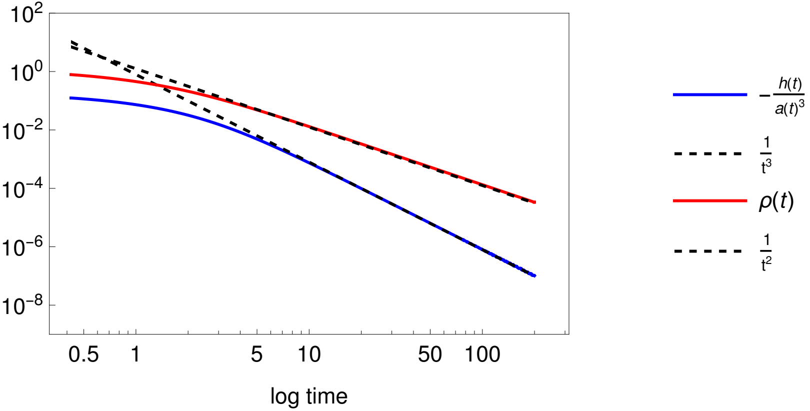
Assuming $a(t) \ll 1$ (early-time universe)

$$\frac{\dot{a}}{a} = \frac{\sqrt{2}}{3\sqrt{3}\sqrt{\beta}} + O(a^{3/2}) \qquad \lambda = \frac{\sqrt{2}}{3\sqrt{3}\sqrt{\beta}}$$

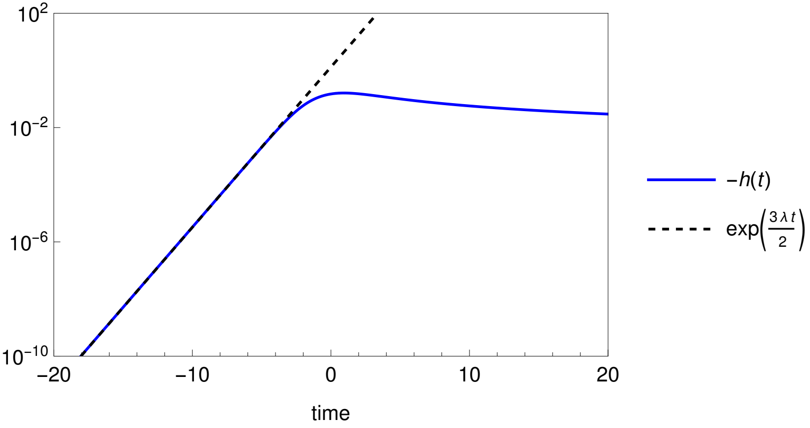
Neat cosmological application – results



Neat cosmological application – results



Neat cosmological application – results



Brief summary

1. There are (too) many modified theories of gravity.
2. Various different routes of changing the original theory of General Relativity can yield completely different, almost unrelated, theories compare bosonic string theory with the teleparallel equivalent of GR.
3. Non-minimal matter couplings allow for even more contrived models, the possibilities are truly endless!
4. One can create a framework in which it is possible to study a large variety of different theories. In this framework one can understand how all these theories are related.
5. Once this framework is set up, very simple models can yield great results.

Thank you!

‘Modified gravity: a unified approach to metric-affine models’
Journal of Mathematical Physics 64 (2023) 082505 [2301.11051 gr-qc]; 30 pages
Jointly with E. Jensko doi.org/10.1063/5.0150038

‘Modified gravity: a unified approach’
Physical Review D104 (2021) 024010 [2103.15906 gr-qc]; 34 pages
Jointly with E. Jensko doi:10.1103/PhysRevD.104.024010