Semi-symmetric metric gravity

from the Friedmann-Schouten geometry with torsion to dynamical dark energy models

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2. Some historical remarks
3. Geometrical preliminaries
4. Proposed theory and its cosmological applications
5. Thermodynamical interpretation
6. Non-static Einstein manifolds with torsion
7. Semi-symmetric Galilei manifolds
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Definition 1.1. Relativistic spacetime

A **relativistic spacetime** is a pair (M, g) consisting of

(*i*) a smooth 4-dimensional manifold M,

(*ii*) an at least piecewise C^2 (but usually smooth) Lorentzian metric g of signature $(-,+,+,+)$.

Remark: Note that the second condition puts a topological restriction on M, namely $\chi(M) = 0$.

Theorem 1.2. Existence and uniqueness of the Levi-Civita connection

On a relativistic spacetime (M, g) there exists a unique connection $\overset{\circ}{\nabla}$, which satisfies

$$\overset{\circ}{\nabla} g = 0, \quad \text{and} \quad \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X, Y] = 0 \quad \text{for all} \quad X, Y \in \Gamma(TM).$$

Einstein's gravitational dynamics are given by

$$\overset{\circ}{R}_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} \overset{\circ}{R} + \Lambda g_{\nu\sigma} = \kappa T_{\nu\sigma}, \quad \text{where} \quad \kappa = \frac{8\pi G}{c^4} \approx 2.07 \times 10^{-43} N^{-1}.$$

One usually assumes a more rigid structure on M, namely

1. M is orientable, or equivalently allows for a globally defined volume form,
2. M has no closed timelike curves,
3. M is connected,
4. M is globally hyperbolic.

Remark: The last condition is equivalent to any of the following conditions

- the existence of a Cauchy surface,
- the existence of a $SL(2, \mathbb{C})$ spin-structure,
- $M \cong R \times \Sigma$, where Σ is a 3-dimensional oriented manifold.

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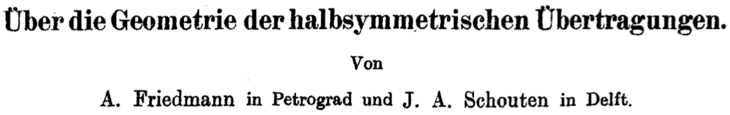
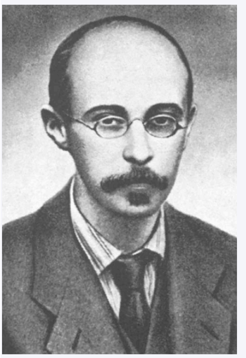


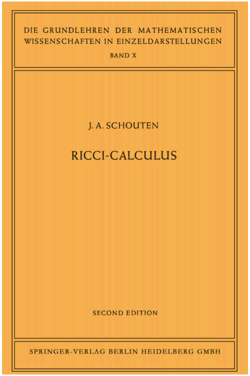
Über die Geometrie der halbsymmetrischen Übertragungen.

Von

A. Friedmann in Petrograd und J. A. Schouten in Delft.

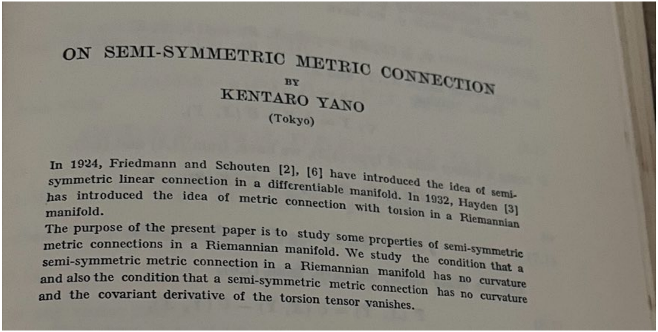


Physical motivation



III 4,1[2]). A man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North pole. Prove that this displacement is semi-symmetric and metric and compute S_λ.[3]) During the mathematical congress in Moscow in 1934 one evening some of us invented the "Moscow displacement". The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. Now let a person walk in the street always facing the Kremlin. Prove that also this displacement is semi-symmetric and metric and compute S_λ.

Mathematical legacy



ON SEMI-SYMMETRIC METRIC CONNECTION

BY

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(Tokyo)

In 1924, Friedmann and Schouten [2], [6] have introduced the idea of semi-symmetric linear connection in a differentiable manifold. In 1932, Hayden [3] has introduced the idea of metric connection with torsion in a Riemannian manifold.

The purpose of the present paper is to study some properties of semi-symmetric metric connections in a Riemannian manifold. We study the condition that a semi-symmetric metric connection in a Riemannian manifold has no curvature and also the condition that a semi-symmetric metric connection has no curvature and the covariant derivative of the torsion tensor vanishes.

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A general affine connection is fully characterized by its torsion and non-metricity

$$Q_{\mu\nu\rho} = -\nabla_\mu g_{\nu\rho}, \quad T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho}.$$

With the help of these, the affine connection can be decomposed as

$$\Gamma^\mu{}_{\nu\rho} = \gamma^\mu{}_{\nu\rho} + \frac{1}{2}g^{\mu\lambda}\left(Q_{\nu\rho\lambda} + Q_{\rho\lambda\nu} - Q_{\lambda\nu\rho}\right) - \frac{1}{2}g^{\mu\lambda}(T_{\rho\nu\lambda} - T_{\lambda\rho\nu} + T_{\nu\rho\lambda}).$$

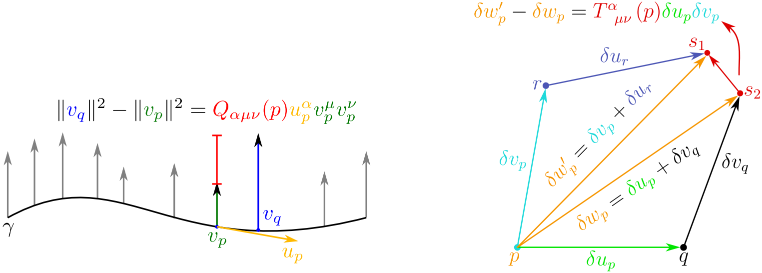
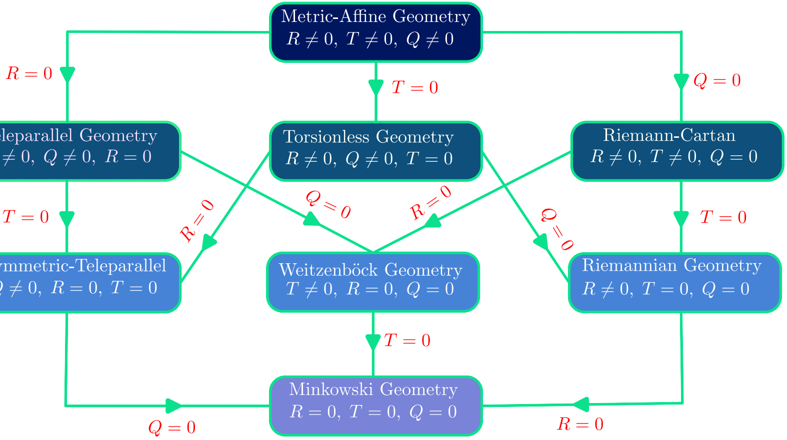


Figure 1: The effects of non-metricity (left panel) and torsion (right panel) are depicted. Non-metricity changes the lengths of vectors, while torsion measures to what extent the parallelogram law fails infinitesimally.


Metric-Affine Geometry
$R \neq 0,\ T \neq 0,\ Q \neq 0$
$R = 0$
$T = 0$
$Q = 0$
Teleparallel Geometry
$T \neq 0,\ Q \neq 0,\ R = 0$
Torsionless Geometry
$R \neq 0,\ Q \neq 0,\ T = 0$
Riemann-Cartan
$R \neq 0,\ T \neq 0,\ Q = 0$
$T = 0$
$R = 0$
$Q = 0$
$R = 0$
$Q = 0$
$T = 0$
Symmetric-Teleparallel
$Q \neq 0,\ R = 0,\ T = 0$
Weitzenböck Geometry
$T \neq 0,\ R = 0,\ Q = 0$
Riemannian Geometry
$R \neq 0,\ T = 0,\ Q = 0$
$T = 0$
Minkowski Geometry
$R = 0,\ T = 0,\ Q = 0$
$Q = 0$
$R = 0$

Vectorial geometries

1. **Weyl geometry**: $Q_{\mu\nu\rho} = w_\mu g_{\nu\rho}$, $T = 0$ - special case of torsionless geometry.
2. **Semi-symmetric metric geometry**: $Q = 0$, $T^\mu{}_{\nu\rho} = \pi_\rho \delta^\mu_\nu - \pi_\nu \delta^\mu_\rho$ - special case of Riemann-Cartan geometry.
3. **Schrödinger geometry**: $Q^\alpha{}_{\mu\nu} = g_{\mu\nu}\pi^\alpha - \frac{1}{2}\left(\delta^\alpha_\mu \pi_\nu + \delta^\alpha_\nu \pi_\mu\right)$, $T = 0$.

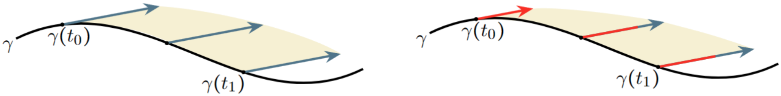


Figure 2: Illustration of the effect of non-metricity on autoparallel transport. On the left panel, one can see a Schrödinger-type non-metricity, which preserves lengths of autoparallely transported vectors, while on the right panel the effect of a general non-metricity is depicted.

Generally, a Schrödinger connection is given by

$$\Gamma^{\mu}{}_{\nu\rho} = \gamma^{\mu}{}_{\nu\rho} + g^{\mu\lambda} Z_{\lambda\nu\rho},$$

where $Z_{\lambda\nu\rho}$ satisfies

$$Z_{\lambda\nu\rho} = Z_{\lambda\rho\nu}, \text{ and } Z_{\lambda\nu\rho} + Z_{\rho\lambda\nu} + Z_{\nu\rho\lambda} = 0.$$

For a semi-symmetric metric connection, we have

$$T^{\mu}{}_{\nu\rho} = \pi_\rho \delta^\mu_\nu - \pi_\nu \delta^\mu_\rho \iff T_{\lambda\nu\rho} = \pi_\rho g_{\lambda\nu} - \pi_\nu g_{\lambda\rho}.$$

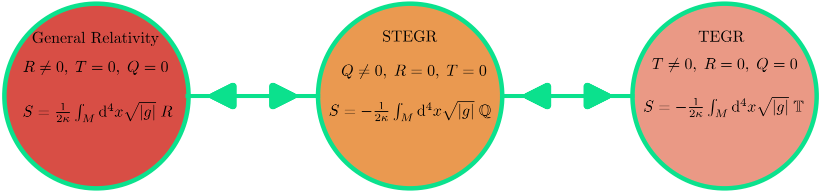
If we symmetrize T accordingly, we obtain a Schrödinger connection

$$Z_{\lambda\nu\rho} = T_{(\nu\rho)\lambda} = \frac{1}{2}\left(\pi_\lambda g_{\nu\rho} - \pi_\rho g_{\nu\lambda} + \pi_\lambda g_{\nu\rho} - \pi_\nu g_{\rho\lambda}\right).$$

In this sense, a **vectorial** Schrödinger connection can be realized by semi-symmetric type of torsion.

Main four reasons

1. Ehlers-Pirani-Schild axiomatization of GR. arXiv:2112.14063
2. Torsion-non-metricity duality in $f(R)$ gravity. arXiv:1810.06602
3. Einstein-Cartan theory. arXiv:0711.1535
4. Non-relativistic limit. arXiv:2308.07100



Definition 3.1. Semi-symmetric metric connection

On a relativistic spacetime (M, g), a connection ∇ is called **semi-symmetric metric** if it is metric compatible and there exists $\pi \in \Gamma(T^*M)$, such that

$$\nabla_X Y - \nabla_Y X - [X, Y] = \pi(Y)X - \pi(X)Y, \forall X, Y \in \Gamma(TM).$$

In this case, torsion can be described globally and locally as

$$T(\omega, X, Y) = \pi(Y)\omega(X) - \pi(X)\omega(Y) \text{ and } T^{\mu}{}_{\nu\rho} = \pi_\rho \delta^\mu_\nu - \pi_\nu \delta^\mu_\rho, \text{ respectively.}$$

Theorem 3.2. Existence and uniqueness of semi-symmetric metric connection

Let (M, g, ∇) be a relativistic spacetime with a semi-symmetric metric connection and denote by $\overset{\circ}{\nabla}$ the Levi-Civita connection. Then

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)P,$$

where P is the dual vector field associated to π, i.e. $g(X, P) = \pi(X)$.

Locally, we have $\Gamma^{\mu}{}_{\nu\rho} = \gamma^{\mu}{}_{\nu\rho} - \pi^\mu g_{\rho\nu} + \pi_\nu \delta^\mu_\rho$.

Theorem 3.3. Yano

Let (M, g, ∇) be a relativistic spacetime with a semi-symmetric connection that is metric-compatible and denote with $\overset{\circ}{\nabla}$ the Levi-Civita connection. Moreover, denote by *Riem* the curvature tensor of the semi-symmetric metric connection ∇ and by $\overset{\circ}{Riem}$ the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$. Then, the following equation

$$\begin{aligned} Riem(\omega, Z, X, Y) = \overset{\circ}{Riem}(\omega, Z, X, Y) &- \omega(S(Y, Z)X) \\ &+ \omega(S(X, Z)Y) - \omega(g(Y, Z)A(X)) \\ &+ \omega(g(X, Z)A(Y)), \end{aligned}$$

is satisfied for all one-forms ω, and vector fields X, Y, Z, where

$$S(X, Y) = \left(\overset{\circ}{\nabla}_X \pi\right)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(P)g(X, Y)$$

and A is a $(1, 1)-$tensor field defined by

$$g(A(X), Y) = S(X, Y).$$

In a local coordinate system, the Riemann tensor of a semi-symmetric metric connection takes the form

$$Riem^{\mu}{}_{\nu\rho\sigma} = \overset{\circ}{Riem}{}^{\mu}{}_{\nu\rho\sigma} - S_{\sigma\nu}\delta^{\mu}_{\rho} + S_{\rho\nu}\delta^{\mu}_{\sigma} - g_{\sigma\nu}S_{\rho\lambda}g^{\lambda\mu} + g_{\rho\nu}S_{\sigma\lambda}g^{\lambda\mu},$$

where the tensor $S_{\sigma\nu}$ is defined as

$$S_{\sigma\nu} = \overset{\circ}{\nabla}_{\sigma}\pi_{\nu} - \pi_{\sigma}\pi_{\nu} + \frac{1}{2}g_{\sigma\nu}\pi_{\lambda}\pi^{\lambda}.$$

We thus immediately obtain that the Ricci tensor and scalar of a semi-symmetric metric connection are given by

(*i*) Ricci tensor: $R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - 2\overset{\circ}{\nabla}_{\nu}\pi_{\mu} + 2\pi_{\nu}\pi_{\mu} - 2g_{\mu\nu}\pi_{\lambda}\pi^{\lambda} - g_{\mu\nu}\overset{\circ}{\nabla}_{\lambda}\pi^{\lambda}$.

(*ii*) Ricci scalar: $R = \overset{\circ}{R} - 6\overset{\circ}{\nabla}_{\alpha}\pi^{\alpha} - 6\pi_{\alpha}\pi^{\alpha}$.

Upon symmetrization, there is a formal analogy with the Weyl geometry, where we have

(*i*) Ricci tensor:

$$R_{(\mu\nu)} = \overset{\circ}{R}_{\mu\nu} + g_{\mu\nu}\overset{\circ}{\nabla}_{\alpha}w^{\alpha} + 3\overset{\circ}{\nabla}_{\nu}w_{\mu} - \overset{\circ}{\nabla}_{\mu}w_{\nu} - 2g_{\mu\nu}w_{\rho}w^{\rho} + 2w_{\mu}w_{\nu}.$$

(*ii*) Ricci scalar: $R = \overset{\circ}{R} + 6\overset{\circ}{\nabla}_{\alpha}w^{\alpha} - 6w_{\alpha}w^{\alpha}$.

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First of all, we *postulate* that

$$R_{(\nu\sigma)} - \frac{1}{2} g_{\nu\sigma} R = 8\pi T_{\nu\sigma}.$$

Post-Riemannian expansion leads to

$$\mathring{R}_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} \mathring{R} - \mathring{\nabla}_\sigma \pi_\nu - \mathring{\nabla}_\nu \pi_\sigma + 2\pi_\sigma \pi_\nu + 2 g_{\sigma\nu} \mathring{\nabla}_\lambda \pi^\lambda + g_{\nu\sigma} \pi^\rho \pi_\rho = 8\pi T_{\nu\sigma}.$$

Observations

1. In the limit $\pi \to 0$, we recover GR as expected.
2. The torsion vector is fully determined by a vectorial part, and it has contributions to the usual EFE, which could be thought of as a geometric type dark energy.
3. There are no dynamics for torsion! This has to be imposed separately.

Assume homogeneous, isotropic, *spatially flat* FLRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j.$$

The matter content is given by a perfect fluid with energy-momentum tensor

$$T_{\nu\sigma} = \rho u_\nu u_\sigma + p(u_\nu u_\sigma + g_{\nu\sigma}).$$

The problem is taken into account in a comoving frame

$$u_\nu = (-1, 0, 0, 0) \iff u^\nu = (1, 0, 0, 0).$$

The cosmological principle implies

$$\pi_\nu = (-\omega(t), 0, 0, 0) \iff \pi^\nu = (\omega(t), 0, 0, 0).$$

Theorem 4.1. Friedmann equations in semi-symmetric metric gravity

$$3H^2 = 8\pi\rho - 3\omega^2 + 6H\omega,$$

$$2\dot{H} + 3H^2 = -8\pi p + 4H\omega - \omega^2 + 2\dot{\omega}.$$

The previously mentioned interpretation can now be made explicit as

$$3H^2 = 8\pi\rho - 3\omega^2 + 6H\omega = 8\pi\left(\rho + \rho_{eff}\right) = 8\pi\rho_{tot},$$

$$2\dot{H} + 3H^2 = -8\pi p + 4H\omega - \omega^2 + 2\dot{\omega} = -8\pi\left(p + p_{eff}\right) = -8\pi p_{tot},$$

where we have denoted

$$\rho_{eff} = \frac{1}{8\pi}\left(6H\omega - 3\omega^2\right), \quad p_{eff} = -\frac{1}{8\pi}\left(4H\omega - \omega^2 + 2\dot{\omega}\right),$$

while $\rho_{tot} = \rho + \rho_{eff}$, and $p_{tot} = p + p_{eff}$.

We impose the conditions

$$\frac{1}{8\pi}\left(6H\omega - 3\omega^2\right) = \lambda, \quad -\frac{1}{8\pi}\left(4H\omega - \omega^2 + 2\dot{\omega}\right) = \frac{2}{3}K\lambda.$$

To get rid of signs and factors, we redefine $\Lambda = 8\pi\lambda, k = -K$ to obtain

$$3\omega\left(2H - \omega\right) = \Lambda, \quad 4H\omega - \omega^2 + 2\dot{\omega} = \frac{2}{3}k\Lambda,$$

respectively, where k and $\Lambda \geq 0$ are constants. Eliminating H yields the equation

$$2\dot{\omega} + \omega^2 + 2(1-k)\frac{\Lambda}{3} = 0,$$

which admits an analytical solution

$$\omega(t) = \sqrt{\frac{2(k-1)\Lambda}{3}}\tanh\left[\frac{\sqrt{(k-1)\Lambda}}{\sqrt{6}}\left(t - t_0\right)\right].$$

The Hubble function is given by

$$H(t) = \frac{\sqrt{\Lambda}}{2\sqrt{6(k-1)}} \tanh\left[\frac{\sqrt{(k-1)\Lambda}(t-t_0)}{\sqrt{6}}\right] \times \left\{\coth^2\left[\frac{\sqrt{(k-1)\Lambda}(t-t_0)}{\sqrt{6}}\right] + 2(k-1)\right\}.$$

The matter density $8\pi\rho = 3H^2 - \Lambda$ reads

$$8\pi\rho(t) = \frac{\Lambda}{8(k-1)}\left\{\coth\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] - 2(k-1)\tanh\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right]\right\}^2.$$

Pressure can also be obtained analytically

$$8\pi p(t) = \frac{\Lambda}{24(k-1)}\left\{(4k-7)\text{csch}^2\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] + 4(k-1)^2\text{sech}^2\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] + 4k^2 - 4k - 3\right\}.$$

The scale factor of this cosmological model is given by

$$a(t) = a_0 \sinh^{2(k-1)}\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] \times \cosh^{\sqrt{6}}\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right].$$

Large time limits yield

$$\lim_{t\to\infty}\rho(t) = \frac{(3-2k)^2}{8(k-1)}\Lambda, \quad \lim_{t\to\infty}p(t) = \frac{4k^2-4k-3}{24(k-1)}\Lambda.$$

Comments

1. For $k = \frac{3}{2}$ in the large time limit we have $\lim_{t\to\infty} p(t) = \lim_{t\to\infty}\rho(t) = 0$,indicating that the Universe ends in a vacuum state. For other values of k the cosmological evolution ends in constant density and pressure thermodynamic phase.
2. The deceleration parameter can also be computed analytically. In particular, it can be shown that $\lim_{t\to\infty} q(t) = -1$: the Universe ends in a de Sitter type phase.

We assume a linear equation of state

$$P_{eff}(z) = -\sigma(z) r_{eff}(z) - \lambda,$$

and consider CPL parametrization

$$\sigma(z) = \sigma_0 + \sigma_a \frac{z}{1+z},$$

where σ_0 and σ_a are constants.

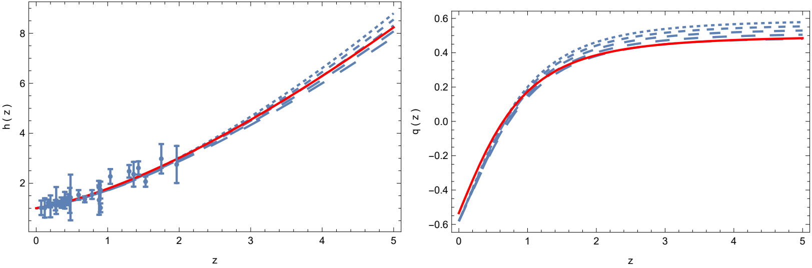

Figure 3: Variations as a function of the redshift z of the dimensionless Hubble function (left panel), and of the deceleration parameter $q(z)$ (right panel) for Model II, for $\lambda = 0.79$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a: $\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

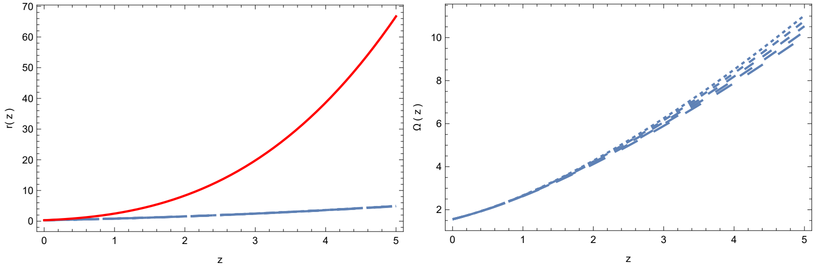

Figure 4: Variations as a function of the redshift z of the dimensionless matter density $r(z)$ (left panel), and of the torsion vector component $\Omega(z)$ (right panel) for Model II, for $\lambda = 0.67$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a:$\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

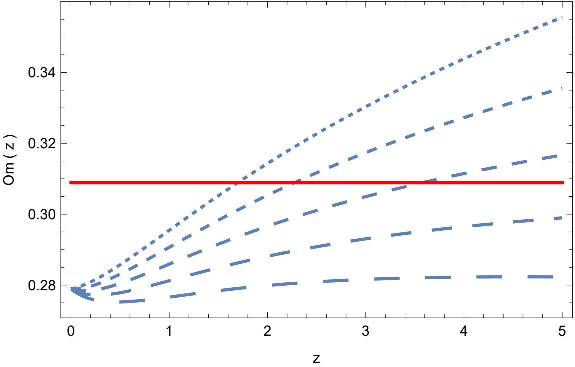

Figure 5: Behavior of the function $Om(z)$ for Model II, for $\lambda = 0.79$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a: $\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

We impose a polytropic equation of state $P_{eff} = K r_{eff}^2$.

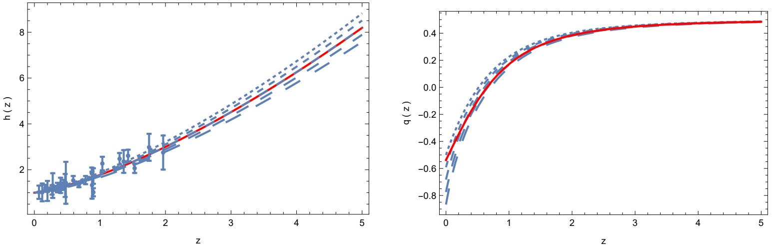

Figure 6: Variations of the dimensionless Hubble function $h(z)$(left panel), and of the deceleration parameter $q(z)$ (right panel) for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve) , $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve),respectively. The observational data for the Hubble function are represented with their error bars, while the red curve depicts the predictions of the ΛCDM model.

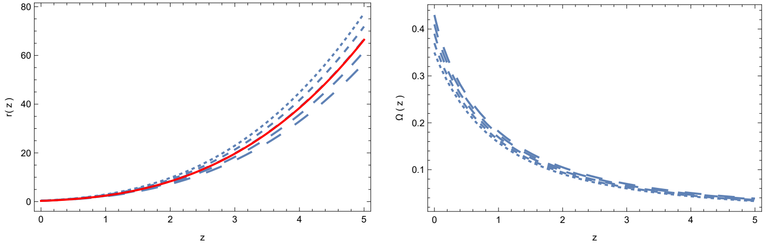

Figure 7: Variations of the dimensionless matter energy density $r(z)$ (left panel), and of the torsion vector $\Omega(z)$ (right panel) for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve) , $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve),respectively. The red curve represents the predictions of the ΛCDM model.

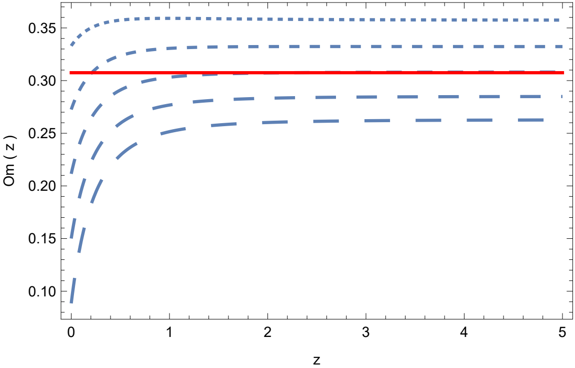

Figure 8: Behavior of the function *Omz* for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve), $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve), respectively. The red curve represents the predictions of the ΛCDM model.

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It can be easily shown that if we impose conservation of matter, the theory reduces to Einstein relativity. Thus, we interpret the non-conservation to be related to particle creation. Hence, we introduce the **thermodynamical quantities**

1. Particle flux $N^\mu \equiv n u^\mu$, where n is the particle number density.
2. Entropy flux vector $S^\mu \equiv s u^\mu = n\sigma u^\mu$, where s is the entropy density, and σ is the entropy per particle.

The particle flux satisfies

$$\nabla_\mu N^\mu = \dot{n} + 3Hn = n\Psi,$$

where Ψ is the particle generation rate. From the second law of thermodynamics we have

$$\nabla_\mu S^\mu = n\dot{\sigma} + n\sigma\Psi \geq 0.$$

The total thermodynamic energy balance equation, $u_\mu \nabla_\nu T^{\mu\nu} = 0$, gives the generalized energy conservation equation in the presence of particle creation

$$\dot{\rho} + 3H(\rho + p + p_c) = 0.$$

The Gibbs law in presence of matter creation is given by

$$nTd\left(\frac{s}{n}\right) = nTd\sigma = d\rho - \frac{\rho + p}{n} dn.$$

From the Friedmann equations, upon some manipulations we can read off

$$p_c = \frac{\omega}{8\pi}\left[2\frac{\dot{H}}{H} - \frac{2\dot{\omega}}{H} + 2H - 2\omega\right].$$

Some easy algebra gives the particle creation rate

$$\Psi = -3H\frac{p_c}{\rho + p} = -\frac{3H\omega}{8\pi(\rho + p)}\left[2\frac{\dot{H}}{H} - \frac{2\dot{\omega}}{H} + 2H - 2\omega\right].$$

The entropy production rate is given by

$$\nabla_\mu S^\mu = -3n\sigma H \frac{p_c}{(\rho + p)} = \frac{3n\sigma H\omega}{8\pi(\rho + p)}\left[2qH + \frac{2\dot{\omega}}{H} + 2\omega\right].$$

We assume the equations of state

$$\rho = \rho(n, T), \quad p = p(n, T).$$

In this case, the temperature evolution takes the form

$$\frac{\dot{T}}{T} = \left(\frac{\partial p}{\partial \rho}\right)_n \frac{\dot{n}}{n} = c_s^2 \frac{\dot{n}}{n} = c_s^2 (\Psi - 3H) = -3c_s^2 H \left(1 + \frac{p_c}{\rho + p}\right), \tag{1}$$

where $c_s^2 = (\partial p/\partial \rho)_n$ is the speed of sound. Hence, in the semi-symmetric metric gravity theory, the time variation of the temperature of the newly created particles is given by

$$\frac{\dot{T}}{T} = 3c_s^2 H \left\{1 + \frac{\omega}{8\pi(\rho + p)}\left[2qH + \frac{2\dot{\omega}}{H} + 2\omega\right]\right\}.$$

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Definition 6.1. Einstein metric

A semi-Riemannian metric g on a smooth manifold M is called an **Einstein metric** if there exists a smooth function $\lambda : M \to \mathbb{R}$, such that

$$\overset{\circ}{R}_{\mu\nu} = \lambda g_{\mu\nu}. \tag{2}$$

Remark: Note that for a general affine connection ∇ equation (2) does not make sense, as $R_{\mu\nu}$ is not symmetric in general. Hence, we will symmetrize in our following definition accordingly.

Definition 6.2. Generalized Einstein manifold

A relativistic spacetime (M, g) equipped with an affine connection ∇ is called a **generalized Einstein manifold** if there exists a smooth function $\lambda : M \to \mathbb{R}$ such that

$$R_{(\mu\nu)} = \lambda g_{\mu\nu}.$$

Proposition 6.3. Characterization of semi-symmetric Einstein manifolds

Let (M, g) be a relativistic spacetime equipped with a semi-symmetric metric connection. Then the following are equivalent:

(i) (M, g) is a generalized Einstein-manifold.

(ii) $\mathring{R}_{\mu\nu} - \mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = \frac{1}{4} g_{\mu\nu} \mathring{R}$.

Trivially, a relativistic spacetime (M, g) equipped with a semi-symmetric connection and an Einstein metric is a generalized Einstein manifold iff

$$\mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = 0.$$

Theorem 6.4. Existence of Einstein manifolds with torsion

There exists a generalized Einstein manifold (M, g), where g is neither an Einstein metric, nor static.

Fix $M = \mathbb{R}^4$ and equip with with the metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j,$$

where we assume that $\frac{\dot{a}}{a} = H_0$ is a non-zero constant. Moreover, we equip $\left(\mathbb{R}^4, ds^2\right)$ with a semi-symmetric connection. The constructed tuple is a generalized Einstein manifold iff

$$\mathring{R}_{\mu\nu} - \mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = \frac{1}{4} g_{\mu\nu} \mathring{R}$$

is satisfied. As the metric possesses high symmetry, we choose

$$\pi_\mu = (\psi(t), 0, 0, 0) \iff \pi^\mu = (-\psi(t), 0, 0, 0).$$

Hence, we have to satisfy the following system of differential equations for $\psi(t)$

$$-3\frac{\ddot{a}}{a} - \dot{\psi} - \dot{\psi} + 2\psi^2 - \frac{1}{2}\left(-\dot{\psi} - 3\frac{\dot{a}}{a}\psi\right) - \frac{1}{2}\psi^2 = -\frac{6}{4}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right),$$

$$a\ddot{a} + 2\dot{a}^2 + 2a\dot{a}\psi + \frac{1}{2}a^2\left(-\dot{\psi} - 3\frac{\dot{a}}{a}\psi\right) + \frac{1}{2}a^2\psi^2 = \frac{6}{4}a^2\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right).$$

By introducing the Hubble parameter $H_0 = \frac{\dot{a}}{a}$, we obtain

$$-\frac{3}{2}\dot{H}_0 - \frac{3}{2}H_0^2 + \frac{3}{2}H_0^2 - \frac{3}{2}\dot{\psi} + \frac{3}{2}\psi^2 + \frac{3}{2}H_0\psi = 0,$$

$$-\frac{1}{2}\dot{H}_0 - \frac{1}{2}H_0^2 + \frac{1}{2}H_0^2 + \frac{1}{2}H_0\psi - \frac{1}{2}\dot{\psi} + \frac{1}{2}\psi^2 = 0.$$

Our assumption that H_0 is constant implies

$$-\frac{3}{2}\dot{\psi} + \frac{3}{2}\psi^2 + \frac{3}{2}H_0\psi = 0,$$

$$\frac{1}{2}H_0\psi - \frac{1}{2}\dot{\psi} + \frac{1}{2}\psi^2 = 0.$$

We can see that the two equations are identical. Hence, we can solve for example the first one, i.e.

$$\dot{\psi} - \psi^2 - H_0\psi = 0,$$

which is a Bernoulli type differential equation. The solution is given by

$$\psi(t) = -\frac{H_0 e^{H_0(t_0+t)}}{e^{H_0(t_0+t)} - 1}.$$

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Definition 7.1. Galilei manifold

A Galilei manifold (M, τ, h) is a differentiable manifold of dimension $n + 1$ together with a nowhere vanishing clock-form $\tau \in \Gamma(T^* M)$ and a spatial metric h, which is positive semidefinite of rank n and satisfies

$$\tau_\mu h^{\mu\nu} = 0.$$

For a torsion-free connection ∇, the Poisson equation can be recast as

$$R_{\mu\nu} = 4\pi G\rho\tau_\mu\tau_\nu,$$

which are called the **Newton-Cartan** field equations.
Interestingly, here torsion and non-metricity are not independent. For a connection compatible with (τ, h), we must have

$$T(\tau, X, Y) = d\tau(X, Y).$$

This means, that in particular, for a semi-symmetric connection, we must have

$$d\tau(X, Y) = \tau(X)\pi(Y) - \tau(Y)\pi(X) = (\tau \wedge pi)(X, Y).$$

In this way, we obtain a semi-symmetric Galilei connection, whose Christoffel symbols can be explicitly calculated.

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Further research possibilities

Some directions of possible future work are:

1. Semi-symmetric $f(R)$ gravity.
2. Black hole solutions in the proposed geometry.
3. Wormhole solutions in the proposed geometry.
4. The study of the early universe: inflationary models.
5. Data analysis, MCMC testing of semi-symmetric metric gravity.
6. Non-relativistic limits, torsional Galilei manifolds, corrections to the Poisson equation.
7. Galaxy rotation curves in this modified gravity theory.

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