Anomalous Luttinger (Tolman-Ehrenfest) equivalence between temperature and curved spacetime: From black hole's atmosphere to thermal quenches

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in collaboration with

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B. Bermond, D. Carpentier, A. Grushin, M.C. 2021, ArXiv:2206.08784

Goals

1) Revisit classical relation between local equilibrium temperature and gravitational potential in the presence of gravitational anomalies

— Classical Tolman-Ehrenfest relation and Luttinger trick (*)

— Anomaly corrections to the Tolman-Ehrenfest/Luttinger relations

2) Uncover measurable effects: non-equilibrium thermal systems
black hole atmospheres, heat quenches and Floquet states (in ArXiv:2206.08784)

(*) PHYSICAL REVIEW VOLUME 135, NUMBER 6A 14 SEPTEMBER 1964

Theory of Thermal Transport Coefficients*

J. M. LUTTINGER
Department of Physics, Columbia University, New York, New York

Footnote 7: "In fact, if the gravitational field didn't exist, one could invent one for the purposes of this paper."

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recent review:

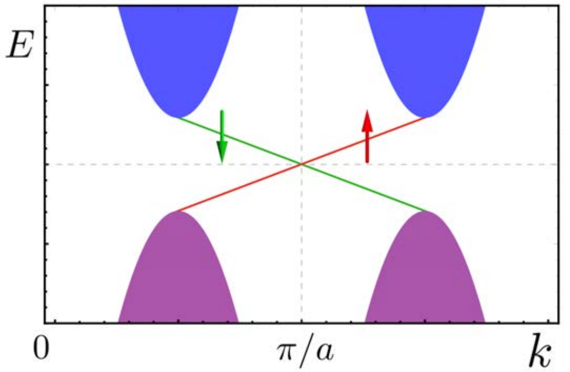
Thermal transport, geometry, and anomalies

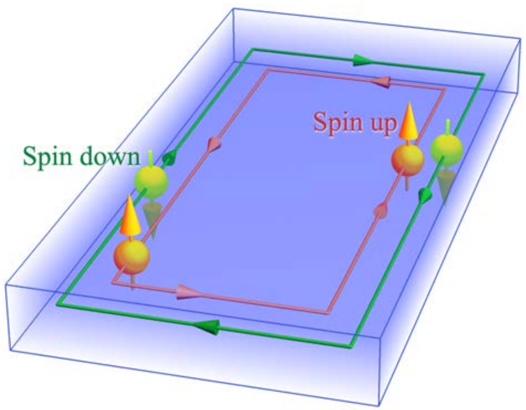
Maxim N. Chernodub [a], Yago Ferreiros [b], Adolfo G. Grushin [c], Karl Landsteiner [d], María A.H. Vozmediano [e,*]

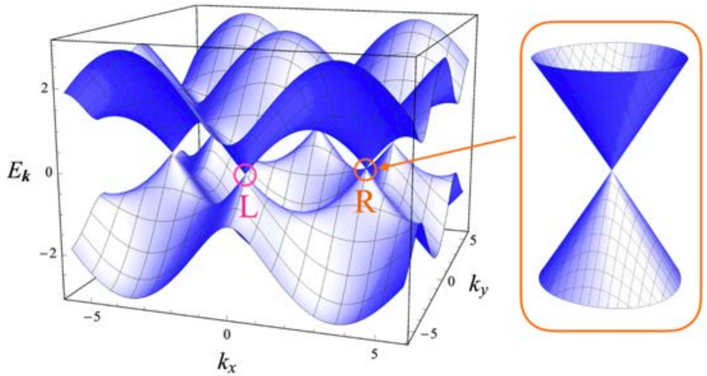
Physics Reports 977 (2022) 1

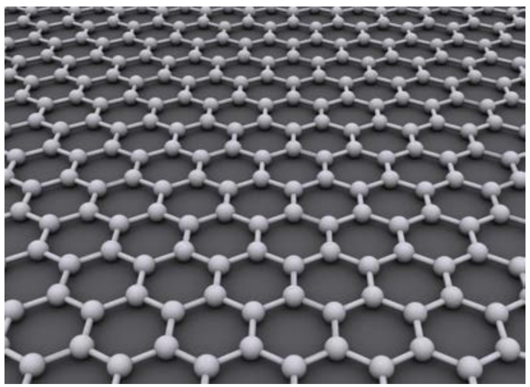
Relativistic physics in crystals $E = \hbar v_F |\vec{k}|$

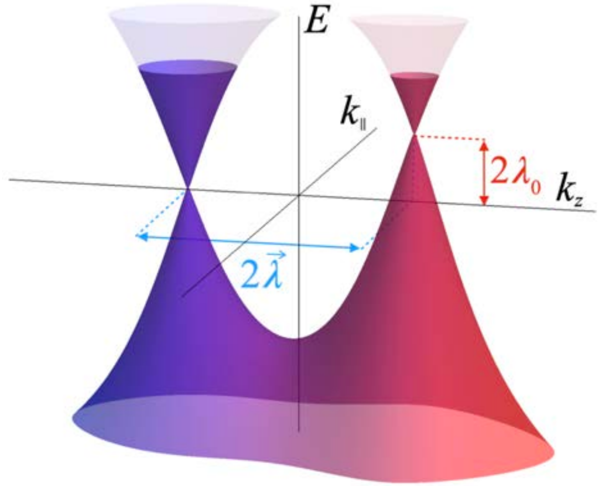
$D = 1+1$	$D = 2+1$	$D = 3+1$
topological insulators	**graphene**	**Weyl/Dirac semimetals**

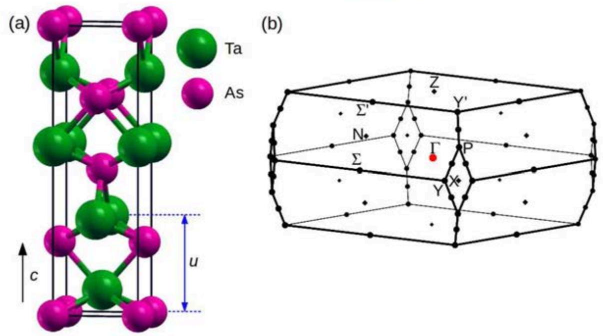










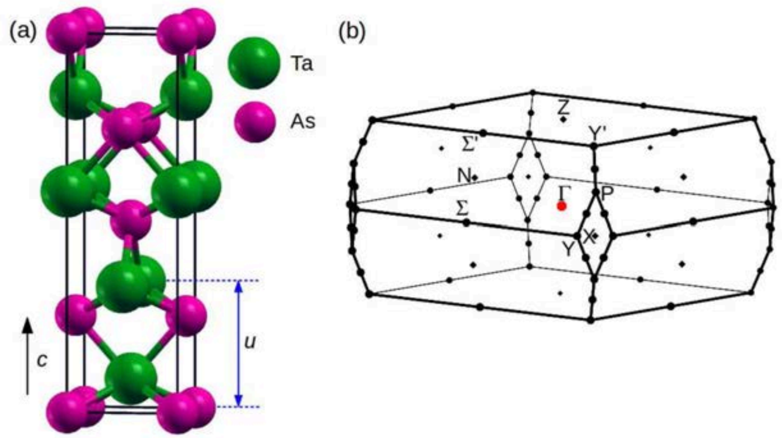


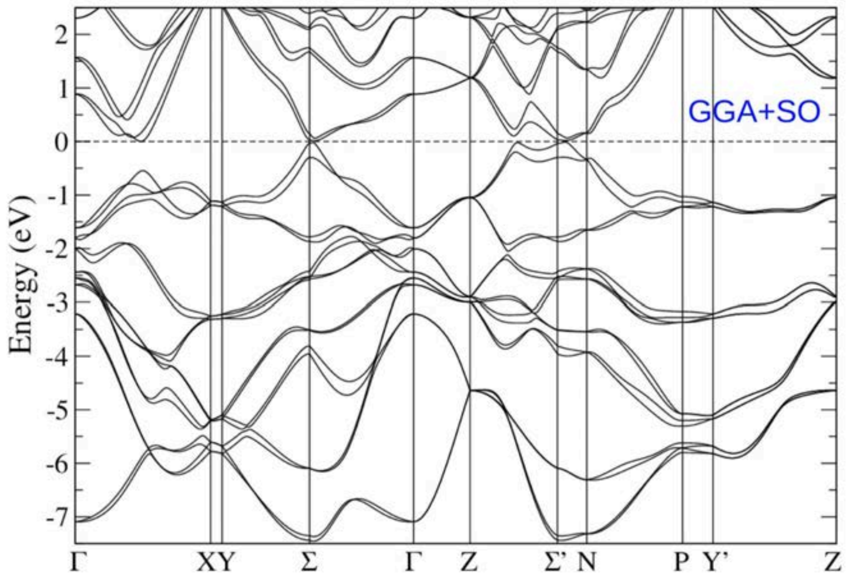
linear dispersion = relativistic particle spectrum

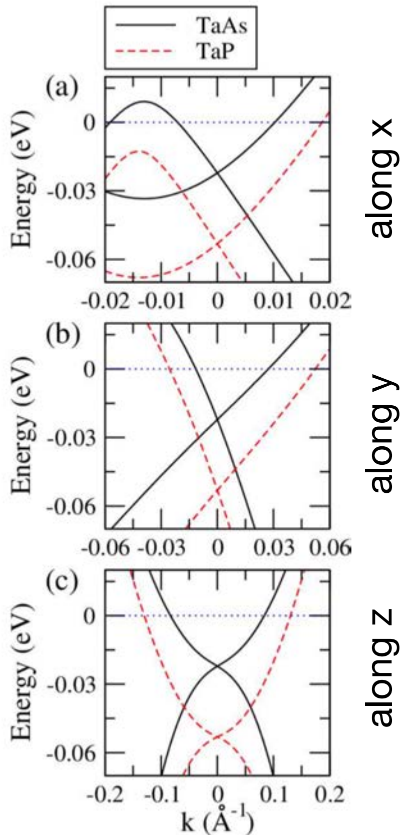
$$v_F \simeq 10^6 \,\mathrm{m/s} \sim \frac{c}{300}$$

Relativistic physics in 3d crystals

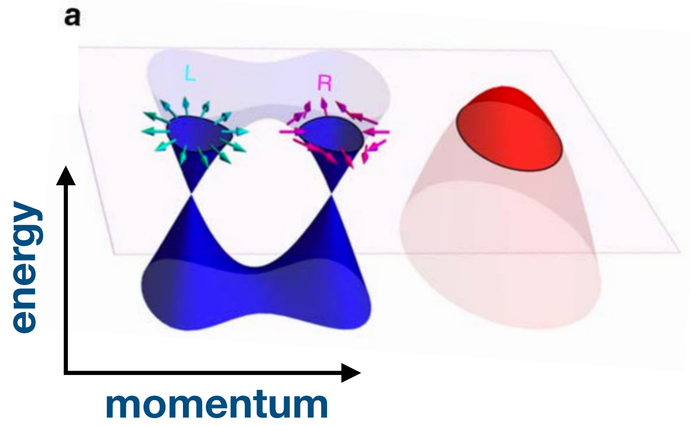
TaAs as an example (of a Weyl semimetal)

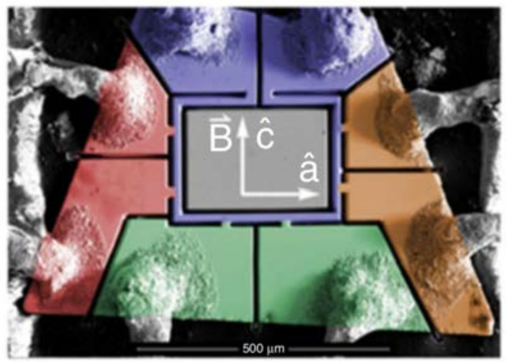






Chi-Cheng Lee et al.
Phys. Rev. B 92, 235104 (2015)





from Nat. Comm. 9, 2217 (2018)
B. J. Ramshaw et al.

Massless (gapless) Dirac fermions

A generic system in particle physics, cosmology, solid state …

Covariant formulation (quantum field theory)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

$$\not{D} = \gamma^\mu D_\mu \qquad D_\mu = \partial_\mu + ieA_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

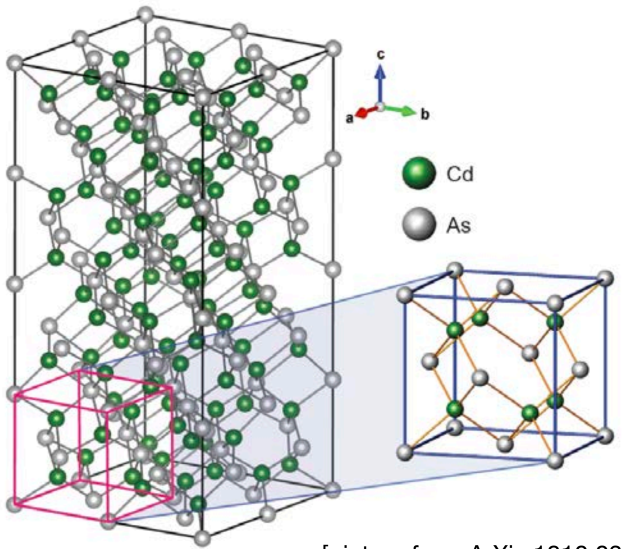
Dirac semimetals (solid state):

$$\bar{\psi}\left[i\gamma^0 \hbar \frac{\partial}{\partial t} + v_F \boldsymbol{\gamma}(i\hbar\boldsymbol{\nabla} - e\boldsymbol{A})\right]\psi$$

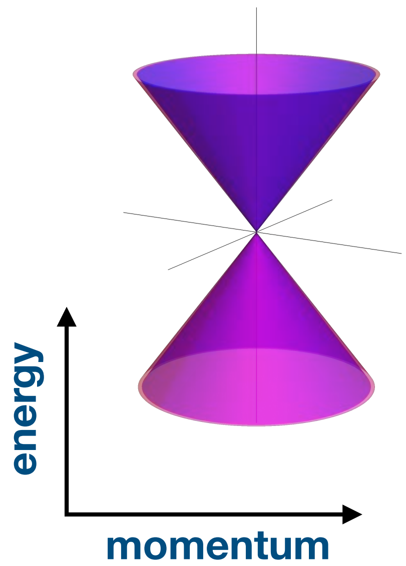
Fermi velocity:

$$v_F \simeq 10^6\,\mathrm{m/s} \sim \frac{c}{300}$$

example: Cd_3As_2 (Dirac semimetal)



[picture from ArXiv:1810.03726]



band structure:

relativistic spectrum of electronic quasiparticles close to the Fermi surface around an isolated Dirac cone

Effective low energy description around band crossings in 3D crystals.

Classical symmetries in (3+1)d

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

Vector

$$\psi \to e^{i\omega_V} \psi$$

local/gauge symmetry

vector current is classically conserved

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \qquad \partial_\mu j_V^\mu = 0$$

Axial

$$\psi \to e^{i\omega_5 \gamma^5} \psi$$

global symmetry (no axial gauge field)

axial current is classically conserved

$$j_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi \qquad \partial_\mu j_A^\mu = 0$$

Scale

(also called “conformal”)

global scale transformations

$$x \to \lambda^{-1} x, \qquad A_\mu \to \lambda A_\mu, \qquad \psi \to \lambda^{3/2}\psi$$

Dilatation current is classically conserved

$$j_D^\mu = T^{\mu\nu} x_\nu \qquad \partial_\mu j_D^\mu \equiv T^\mu_{\ \mu} \equiv 0$$

in (3+1)d

$$(T^\mu_\mu)_{\text{cl}} \equiv 0$$

Energy-Momentum tensor

$$T^{\mu\nu} = -F^{\mu\alpha} F^\nu_{\ \alpha} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + \frac{i}{2}\bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - \eta^{\mu\nu}\bar{\psi} i \not{D}\psi$$

Classical scale symmetry

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi$$

Vector $\psi \to e^{i\omega_V}\psi$ — vector current is classically conserved

local/gauge symmetry $\quad j_V^\mu = \bar{\psi}\gamma^\mu\psi \quad \partial_\mu j_V^\mu = 0$

We are interested in anomalies generated in curved space time, related to the energy-momentum tensor

Axial $\psi \to e^{i\omega_5\gamma^5}\psi$ — axial current is classically conserved

Terminology: “gravitational anomaly” → quantum fluctuations violate some of classical properties of energy-momentum tensor.

global symmetry (no axial gauge field) $\quad j_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi \quad \partial_\mu j_A^\mu = 0$

Scale (also called “conformal”)

global scale transformations

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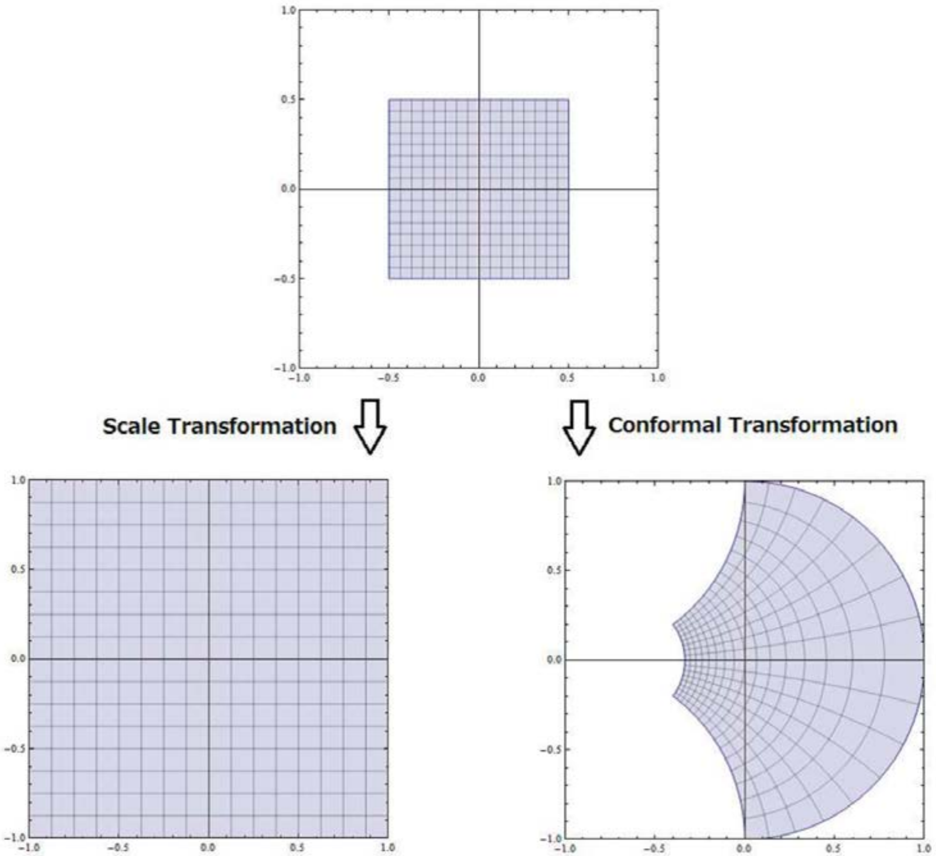
$$T^{\mu\nu} = -F^{\mu\alpha}F^\nu_{\ \alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{i}{2}\bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - \eta^{\mu\nu}\bar{\psi}i\not{D}\psi$$

Terminology



Yu Nakayama,
Phys.Rept. 569 (2015)

Mathematics:
Scale symmetry ≠ conformal symmetry

Physics:
Scale symmetry =conformal symmetry

Often used interchangeably:

Scale anomaly
Conformal anomaly
Trace anomaly
Weyl anomaly

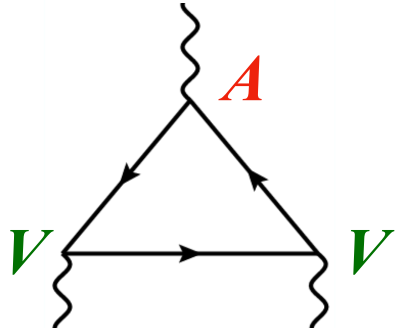
There is no known example of scale invariant but non-conformal field theories in d = 4 dimension under the assumptions of
(1) unitarity,
(2) Poincaré invariance (causality),
(3) discrete spectrum in scaling dimension,
(4) existence of scale current and
(5) unbroken scale invariance in the vacuum.

Zoo of anomalies

(three out of six triangular vertices)

electromagnetic background

Axial



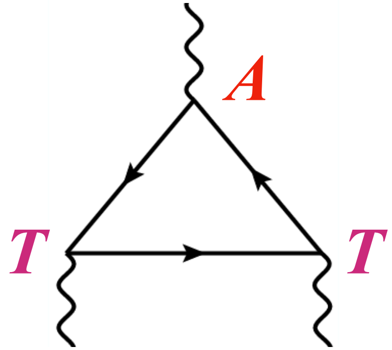
$$\partial_\mu j^\mu_A = -\frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}$$

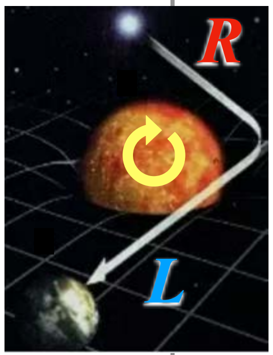
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

gravitational/curved background

Mixed

(axial-gravitational anomaly)





$$\partial_\mu j^\mu_A = -\frac{1}{384\pi^2} R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$$

Riemann tensor (curved spacetime)

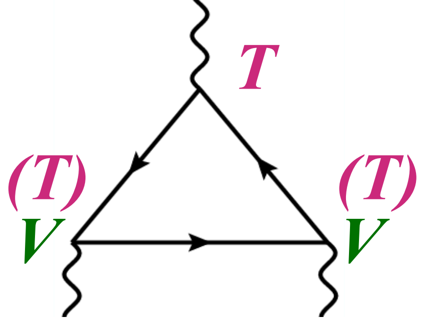
$$\tilde{R}^{\mu\nu\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu\gamma\lambda}R_{\gamma\lambda}{}^{\alpha\beta}$$

electromagnetic background

Conformal

(scale, trace)

not one-loop exact



$$\partial_\mu j^\mu_D = T^\alpha_\alpha$$

beta function

$$\langle T^\mu_{\ \mu}\rangle = \frac{\beta(e)}{2e} F_{\mu\nu}F^{\mu\nu}$$

Currents

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi, \qquad j^\mu_A = \bar{\psi}\gamma^5\gamma^\mu\psi \qquad j^\mu_D = T^{\mu\nu}x_\nu$$

Vector — Axial — Dilatation

Full list: AVV, ATT, TVV, TAA, AAA, TTT (not counting torsion!)

Energy-Momentum tensor

$$T^{\mu\nu} = -F^{\mu\alpha}F^\nu_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{i}{2}\bar{\psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - \eta^{\mu\nu}\bar{\psi}i\not{D}\psi$$

Conformal (trace) anomaly in 3+1 dimensions

Conformal anomaly in curved space-time in electromagnetic field background

$$\langle T^{\mu}_{\ \mu} \rangle \equiv -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta S_{\rm anom}}{\delta g_{\mu\nu}}$$

anomalous action which generates anomaly

$$= -\frac{1}{4}\Big[\underbrace{bC^2 + b'\Big(E - \frac{2}{3}\Box R\Big)}_{\text{gravitational (curved space-time) background}} + \underbrace{cF_{\mu\nu}F^{\mu\nu}}_{\text{electromagnetic background}}\Big]$$

M. Duff, Nucl.Phys.B 125, 334 (1977)

electromagnetic background

gravitational (curved space-time) background

$$C^2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}$$

Weyl tensor squared

$$E = {}^*R_{\mu\nu\alpha\beta}{}^*R^{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$${}^*R_{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\mu'\nu'}R^{\mu'\nu'}{}_{\alpha\beta}$$

Euler (topological) density

Couplings (one Dirac fermion)

$$b = \frac{1}{320\pi^2}, \qquad b' = -\frac{11}{5670\pi^2}, \qquad c = -\frac{e^2}{24\pi^2}$$

comes from beta function

Conformal (trace) anomaly in 3+1 dimensions

Conformal anomaly in curved space-time in electromagnetic field background

weak effects (quadratic in *R* !)

not important term

we do not consider electromagnetic effects here

$$\langle T^{\mu}_{\ \mu} \rangle \equiv -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta S_{\text{anom}}}{\delta g_{\mu\nu}}$$

$$= -\frac{1}{4}\left[\underbrace{bC^2 + b'\left(E - \frac{2}{3}\Box R\right)}_{\text{gravitational (curved space-time) background}} + \underbrace{\cancel{cF_{\mu\nu}F^{\mu\nu}}}_{\text{electromagnetic background}}\right]$$

M. Duff, Nucl.Phys.B 125, 334 (1977)

electromagnetic background

gravitational (curved space-time) background

$$C^2 = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$$
$$\equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}$$

Weyl tensor squared

$$E = {}^*R_{\mu\nu\alpha\beta}{}^*R^{\mu\nu\alpha\beta}$$
$$\equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$${}^*R_{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\mu'\nu'}R^{\mu'\nu'}{}_{\alpha\beta}$$

Euler (topological) density

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$$b = \frac{1}{320\pi^2}, \qquad b' = -\frac{11}{5670\pi^2}, \qquad c = -\frac{e^2}{24\pi^2}$$

comes from beta function

Conformal (trace) anomaly in 1+1 dimensions

Example: Dirac fermions in 1+1 dimensions

Lagrangian

$$\mathcal{L}_g = \frac{i}{2} e_a^\mu(x) \left(\bar{\psi} \gamma^a \overleftrightarrow{\partial}_\mu \psi \right)$$

metric

$$g_{\mu\nu}(x) = \eta_{ab} e_\mu^a(x) e_\nu^b(x)$$

spin connection

$$\omega_\mu = -\frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta} e_\mu^a (\partial_\alpha e_\beta^b) \eta_{ab}$$

energy-momentum tensor

$$T_{\mu\nu}(x) \equiv \frac{1}{2|\det(e_c^\gamma)|} \left[e_\nu^a \frac{\delta S}{\delta e_a^\mu(x)} + (\mu \leftrightarrow \nu) \right]$$

$$= \frac{i}{4} e_\mu^a(x) e_\nu^b(x) \left[\bar{\psi} \left(\gamma_a \overleftrightarrow{\partial}_b + \gamma_b \overleftrightarrow{\partial}_a \right) \psi \right]$$

classically traceless

scalar curvature

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = \frac{\sqrt{-g}}{4} \epsilon_{\mu\nu} R$$

units $\hbar = c = k_B = 1$

The conformal anomaly in (1+1) d

$$\langle T^\mu{}_\mu \rangle = \frac{C}{24\pi} R$$

the central charge (number of species of Dirac fermions)

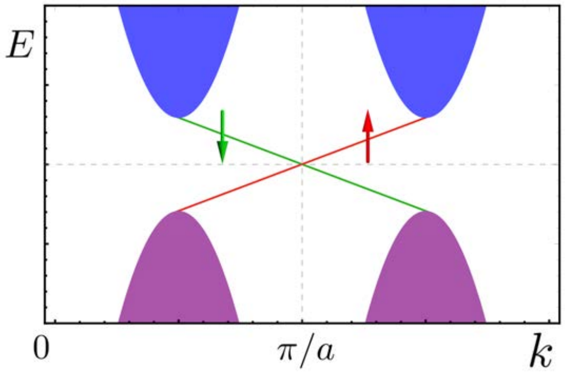
$$C = N_D$$

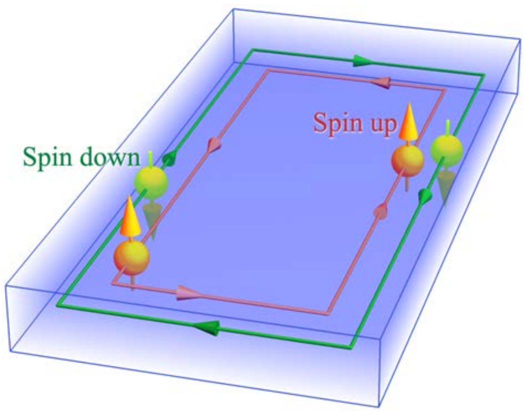
linear in the scalar curvature

parametrically large and simple

Gravitational anomalies

$D = 1+1$	$D = 2+1$	$D = 3+1$
topological insulators	**graphene**	**Weyl/Dirac semimetals**
large and interesting!	**do not exist (in the bulk)**	**exist but weak**





Gravitational anomalies in 1+1 dimensions

Classical

- Conformal/Weyl symmetry

$$\mathcal{T}^{\mu}_{\mu} = 0$$

- Diffeomorphism invariance

$$\nabla_{\mu}\mathcal{T}^{\mu}_{\nu} = 0$$

- Lorentz invariance

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu}$$

Quantum

- Conformal **anomaly**

$$\mathcal{T}^{\mu}_{\mu} = C_w \frac{\hbar v_F}{48\pi}\mathcal{R}$$

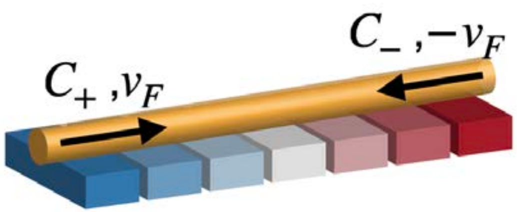
- Einstein **anomaly**

$$\nabla_{\mu}\mathcal{T}^{\mu\nu} = C_g \frac{\hbar v_F}{96\pi}\frac{1}{\sqrt{-det(g)}}\epsilon^{\nu\mu}\nabla_{\mu}\mathcal{R}$$

- Lorentz **anomaly**

$$\mathcal{T}^{\mu\nu} - \mathcal{T}^{\nu\mu} = C_g \frac{\hbar v_F}{48\pi}\frac{1}{\sqrt{-det(g)}}\epsilon^{\mu\nu}\mathcal{R}$$

quantum wire with right and left movers



central charge: $c_w = (C_+ + C_-)/2$

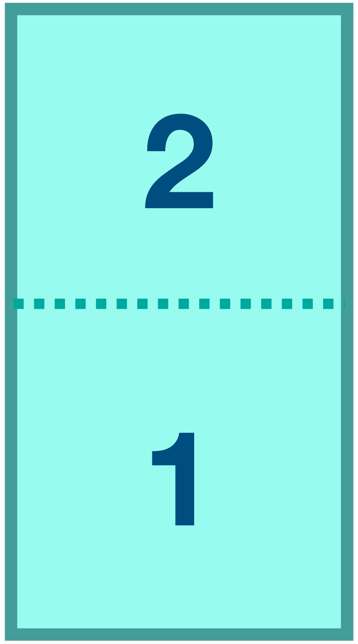
imbalance of movers: $c_g = (C_+ - C_-)/2$

(slide by Baptiste Bermond)

Gravity and temperature? 1. Thermal equilibrium

– Consider a closed system divided arbitrarily into two subsystems

– Assume that we have no gravitational field



– Thermal equilibrium happens when the total entropy reaches its maximum

$$S = S_1 + S_2 \qquad dS_1 + dS_2 = 0$$

– If the quantity of heat leaves the first subsystem, it always enters the second subsystem:

$$dE_1 = -dE \rightarrow dE_2 = dE \rightarrow dS_1/dE_1 = dS_2/dE_2 \rightarrow T_1 = T_2$$

definition of temperature $1/T = \partial S/\partial E$

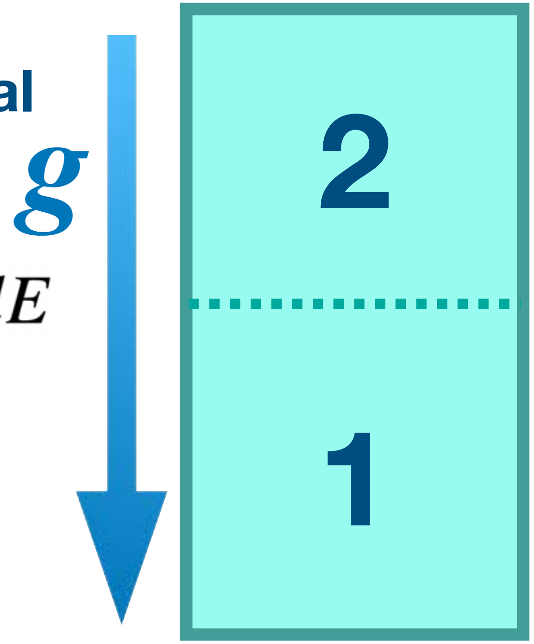
In the absence of gravitational field, the temperature is constant

Gravity and temperature! The Tolman-Ehrenfest law

- In a static gravitational field Φ, the heat quantity dE possesses an inertial mass

$$dm = dE/c^2$$

- the equivalence between inertial and gravitational masses: a quantity of heat has a weight



- When heat leaves the first subsystem, $dE_1 = -dE$ it enters the second subsystem, and performs work agains the gravity (heat = mass):

$$dE_2 = dE + (\Phi_2 - \Phi_1)dm = dE_2(1 + \Delta\Phi/c^2)$$

$$\Delta\Phi = \Phi_2 - \Phi_1$$

change of the gravitational potential

- Entropy maximum

$$dS_1 + dS_2 = 0$$

$$T_1 = T_2$$

- Local temperature

$$T_2 = T_1(1 + \Delta\Phi/c^2)$$

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

Tolman-Ehrenfest law

$$T(x) = T_0/\sqrt{g_{00}(x)}$$

C. Rovelli and M. Smerlak, “Thermal time and Tolman-Ehrenfest effect: temperature as the speed of time” Class. Quant. Grav. 28, 4 (2010)

Tolman-Ehrenfest law → thermal Luttinger

In a static background gravitational field, the temperature of a system in a thermal equilibrium is not uniform:

$$T(\boldsymbol{x})\sqrt{g_{00}(\boldsymbol{x})} = T_0$$

reference temperature

local temperature

static gravitational field

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

R. C. Tolman, “On the Weight of Heat and Thermal Equilibrium in General Relativity,” Phys. Rev. 35, 904 (1930);
R. Tolman and P. Ehrenfest, “Temperature Equilibrium in a Static Gravitational Field,” Phys. Rev. 36, 1791 (1930).

Gravity of Earth (a negligible effect): $$\frac{1}{T}\frac{dT}{dr} = -\frac{g}{c^2} \sim -10^{-18}\mathrm{cm}^{-1}$$

Important for thermal transport (via Kubo formula):

$$\frac{1}{T}\boldsymbol{\nabla}T = -\frac{1}{c^2}\boldsymbol{\nabla}\Phi$$

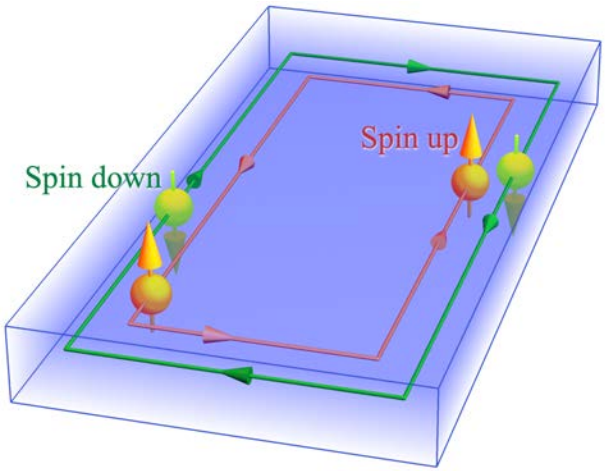
Theory of Thermal Transport Coefficients*

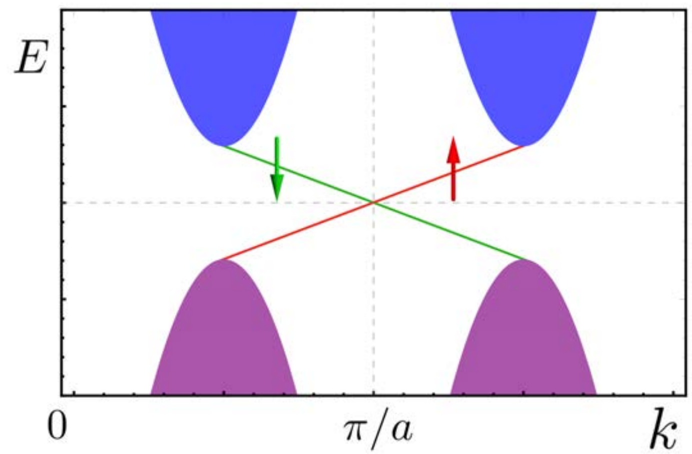
J. M. Luttinger

Department of Physics, Columbia University, New York, New York

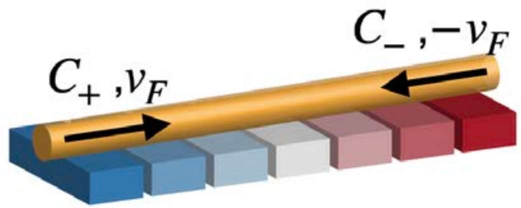
(Received 20 April 1964)

The scale anomaly in 1+1 dimensions, and violation of thermal Luttinger relation





One could think about topologically protected helical edge states in topological insulators, quantum wires



Thermal Luttinger relation

Tolman-Ehrenfest effect:

thermal energy has mass → should feel gravity

$$T(x)\sqrt{g_{00}(x)} = \mathrm{const}$$

R. C. Tolman, "On the Weight of Heat and Thermal Equilibrium in General Relativity," Phys. Rev. 35, 904 (1930);
R. Tolman and P. Ehrenfest, "Temperature Equilibrium in a Static Gravitational Field," Phys. Rev. 36, 1791 (1930)

(thermal) Luttinger relation:

Tolman-Ehrenfest in linear response:

$$g_{00} = 1 + \frac{2\Phi}{c^2} \longrightarrow \frac{1}{T}\boldsymbol{\nabla} T = -\frac{1}{c^2}\boldsymbol{\nabla}\Phi$$

J. M. Luttinger "Theory of Thermal Transport Coefficients," Phys. Rev. 135, A1505 (1964).

Statement: does not work in 1+1 dimensions due to scale anomaly

J. Park, O. Golan, Y. Vinkler-Aviv, A. Rosch, Phys. Rev. B 105, 205419 (2022)
B. Bermond, D. Carpentier, A. Grushin, M.C. 2021, ArXiv:2206.08784

The Tolman-Ehrenfest effect (no anomaly)

metric

$$g_{\mu\nu}(x) = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & -1 \end{pmatrix}$$

energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu}$$

$$T^\mu{}_\nu = \begin{pmatrix} \varepsilon & 0 \\ 0 & -p \end{pmatrix}$$

energy-momentum conservation

$$\nabla_\nu T^\nu{}_\mu \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(T^\nu{}_\mu \sqrt{-g}\right) - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} T^{\alpha\beta} = 0$$

$$g_{00} = 1 + 2\Phi$$

momentum conservation (μ = 1)

$$g_{00} \frac{\partial p}{\partial x} = -\frac{\varepsilon + p}{2} \frac{\partial g_{00}}{\partial x} \xrightarrow[\varepsilon = p]{\text{no anomaly}} \frac{\partial}{\partial x}(g_{00} p) = 0$$

$$\Big\downarrow \; p = \varepsilon = \gamma T^2 \;\; \text{no anomaly}$$

$$\frac{1}{T}\frac{\partial T}{\partial x} + \frac{\partial \Phi}{\partial x} = 0 \xleftarrow{\text{linear response}} T(x)\sqrt{g_{00}(x)} = \text{const}$$

the Luttinger relation

the Tolman-Ehrenfest effect

Trace anomaly in 1+1 dimensions

Tolman-Ehrenfest effect is no more valid in the presence of the trace anomaly

Y. Gim and W. Kim, “A Quantal Tolman Temperature,” Eur. Phys. J. C 75, 549 (2015);
M. Eune, Y. Gim and W. Kim, “Effective Tolman temperature induced by trace anomaly” Eur. Phys. J. C 77, no.4, 244 (2017)

the trace anomaly

$$\langle T^{\mu}_{\ \mu} \rangle = \frac{C}{24\pi} R$$

the central charge (number of species of Dirac fermions)

$$C = N_D$$

scalar curvature

linear response

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

perturbation

$$|h_{\mu\nu}| \ll 1$$

scalar curvature

$$R^{(1)} = \partial_\mu \partial_\nu h^{\mu\nu} - \eta^{\mu\nu} \Box h_{\mu\nu}$$

metric (for a weak field)

$$g_{00} = 1 + 2\Phi + \dots, \qquad |\Phi| \ll 1$$

linearized trace anomaly

$$T^{\mu}_{\ \mu} = \frac{N_D}{12\pi} \frac{\partial^2 \Phi}{\partial x^2}$$

The Tolman-Ehrenfest effect (with anomaly)

pressure

$$p = \gamma T^2 - \frac{1}{2} T^\mu_\mu$$

trace (scale) anomaly

$$\varepsilon - p = T^\mu_\mu$$

energy

$$\varepsilon = \gamma T^2 + \frac{1}{2} T^\mu_\mu$$

---

follows from thermodynamics

$$p = -\left(\frac{\partial F}{\partial V}\right)_T, \qquad E = -T^2 \left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V \qquad p = -f \equiv F/V$$

$$\varepsilon = T^2 \frac{\partial}{\partial T}\frac{p}{T} \equiv p + T^3 \frac{\partial}{\partial T}\frac{p}{T^2} \qquad T^3 \frac{\partial}{\partial T}\frac{p}{T^2} = T^\mu_\mu$$

The Tolman-Ehrenfest effect (with anomaly)

momentum conservation in the presence of anomaly:

$$\frac{\partial}{\partial x}\left(g_{00}T^2\right) = \frac{g_{00}}{2\gamma}\frac{\partial}{\partial x}T^{\mu}_{\ \mu}$$

$$g_{00} = 1 + 2\Phi \qquad T^{\mu}_{\ \mu} = \frac{N_D}{12\pi}\frac{\partial^2\Phi}{\partial x^2}$$

$$\frac{1}{T}\frac{\partial T}{\partial x} + \frac{\partial \Phi}{\partial x} - \frac{N_D}{48\pi\gamma T^2}\frac{\partial^3\Phi}{\partial x^3} = 0$$

The thermal Luttinger relation gets modified!

Modification of the Luttinger relation in one spatial dimension

$$\frac{1}{T}\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\frac{\Phi}{c^2} - \frac{\lambda_T^2}{2}\frac{\partial^3}{\partial x^3}\frac{\Phi}{c^2} = 0$$

(for a weak field)

contribution from
the scale anomaly
1+1 dimensions

thermal length:

$$\lambda_T \equiv \frac{\lambda_T}{2\pi} = \frac{\hbar v_F}{2\pi k_B T}$$

thermal scale breaks the scale symmetry!

The Luttinger relation between temperature and gravitational field is broken by the conformal anomaly!

Modification of the Luttinger relation: simple observations

$$\frac{1}{T}\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\frac{\Phi}{c^2} - \frac{\lambda_T^2}{2}\frac{\partial^3}{\partial x^3}\frac{\Phi}{c^2} = 0$$

- higher-derivative term: spatially non-linear effects?
- presence of “gravitational” zero modes:

$$\Phi(x) = \Phi_+ e^{x/(\sqrt{2}\lambda_T)} + \Phi_- e^{-x/(\sqrt{2}\lambda_T)}$$

inhomogeneous gravitational potential gives homogeneous temperature:

$$T = T_0$$

- relevant “long scale”: $\lambda_T(1\,\mathrm{K}) \simeq 1.2\,\mu\mathrm{m}$

The full, non-linear relation gives us two more new physical scales!

B. Bermond, D. Carpentier, A. Grushin, M.C. 2021, ArXiv:2206.08784

Anomalous Luttinger relation (exact form)

Take the parametrization of the metric: $g_{00} = e^{2\phi}$ **and** $g_{11} = -1$

$$\frac{\partial_x T}{T} = -\partial_x \phi + \lambda_T^2(x)\left[(\partial_x\phi)\,\partial_x^2\phi + \frac{1}{2}\partial_x^3\phi\right]$$

thermal length

$$\lambda_T = \frac{\hbar v_F}{2\pi k_B T}$$

Stress-energy tensor:

(γT^2 is a classical result per d.o.f.)

$$\varepsilon = \frac{C_w}{2}\left[\gamma T_0^2 e^{-2\phi(x)} + \frac{\hbar v_F}{12\pi}\,\partial_x^2\phi + \frac{\hbar v_F}{24\pi}(\partial_x\phi)^2\right]$$ **Energy density**

$$p = \frac{C_w}{2}\left[\gamma T_0^2 e^{-2\phi(x)} - \frac{\hbar v_F}{24\pi}(\partial_x\phi)^2\right]$$ **Pressure**

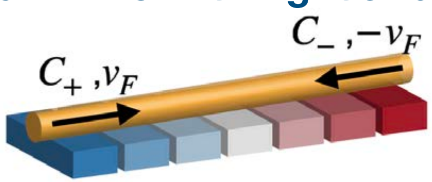
$$J_\varepsilon = v_F^2\Pi = C_g\left[\frac{\pi}{12\hbar}(k_B T_0)^2 e^{-2\phi(x)} + \frac{\hbar v_F^2}{48\pi}\partial_x^2\phi\right]$$ **Energy current**

contributions from the gravitational anomaly

with enforced Lorentz invariance

---

quantum wire with right and left movers



central charge: $c_w = (C_+ + C_-)/2$

imbalance of movers: $c_g = (C_+ - C_-)/2$

Anomalous energy tensor and new scales

Energy-momentum tensor:

$$\mathcal{T} = \mathcal{T}_{\rm cl} + \mathcal{T}_{\rm q}$$

$$c_w = (C_+ + C_-)/2$$
$$c_g = (C_+ - C_-)/2$$

Anomalous contribution:

$$[\mathcal{T}_{\rm q}(x)]^{\mu}_{\ \nu} = \begin{pmatrix} \frac{C_w}{2}\left(\varepsilon_q^{(1)} + \varepsilon_q^{(2)}\right) & \frac{C_g}{2}\sqrt{\frac{f_1}{f_2}}\varepsilon_q^{(2)} \\ -\frac{C_g}{2}\sqrt{\frac{f_2}{f_1}}\varepsilon_q^{(2)} & \frac{C_w}{2}\left(\varepsilon_q^{(1)} - \varepsilon_q^{(2)}\right) \end{pmatrix}$$

new energy scales:

$$\varepsilon_q^{(1)} = \frac{\hbar v_F}{48\pi} R \ ; \qquad \varepsilon_q^{(2)} = \frac{\hbar v_F}{48\pi}(R - 2\overline{R})$$

for the metric

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = f_1(x) v_F^2 dt^2 - f_2(x)\ dx^2$$

$$R = \frac{\partial_x^2 f_1}{f_1 f_2} - \frac{1}{2}\frac{\partial_x f_1}{f_1 f_2}\left[\frac{\partial_x f_1}{f_1} + \frac{\partial_x f_2}{f_2}\right]$$

$$R - 2\overline{R} = R(x) - \frac{1}{f_1(x)}\int_{x_0}^{x} \mathrm{d}y\ R(y)\partial_y f_1(y) = \frac{\partial_x^2 \ln\left(f_1(x)\right)}{f_2(x)} + \frac{1}{2}\partial_x\left(\frac{1}{f_2(x)}\right)\partial_x \ln\left(f_1(x)\right)$$

Signatures of a quantum atmosphere of a black hole

metric: $$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = f_1(x)v_F^2 dt^2 - f_2(x)\, dx^2$$

$$f_1 = 1/f_2 = f$$

Schwarzschild: $f(x) = 1 - x_H/x$

Callan–Giddings–Harvey–Strominger (CGHS): $f(x) = 1 - \exp[-\alpha(x - x_H)]$

close to the horizon x_H :

$$f(x \to x_H^+) \approx 2\kappa c^{-2}(x - x_H)$$ (κ is the surface gravity)

One chiral particle:
(chiral imbalance)

$$C_w = C_g = 1$$

Anomalous scales:

$$\varepsilon_q^{(1)} = \frac{\hbar c}{48\pi}\partial_x^2 f\,; \;\; \varepsilon_q^{(2)} = \frac{\hbar c}{48\pi}\left(\partial_x^2 f - \frac{(\partial_x f)^2}{2f}\right)$$

Hawking temperature

$$k_B T_H = \frac{\hbar}{2\pi c}\kappa$$

Energy current from the black hole (the Hawking radiation):

$$J_\varepsilon = v_F^2 \Pi = C_g \frac{\pi}{12\hbar}\, k_B^2 T^2(x)$$

local temperature of energy flux

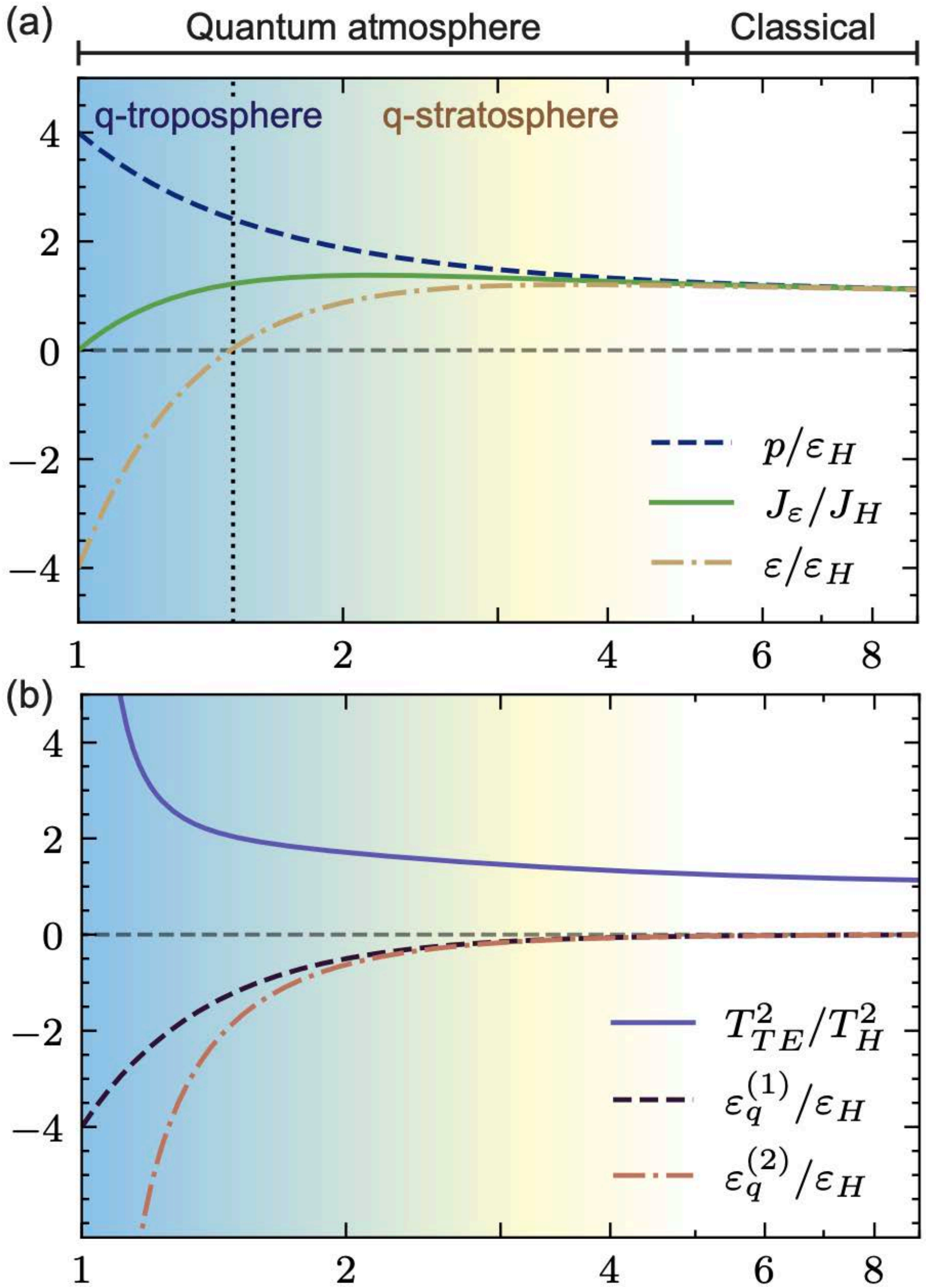
$$k_B^2 T^2(x) = \frac{k_B^2 T_H^2}{f} + \frac{6\hbar c}{\pi}\,\varepsilon_q^{(2)}$$

gravitational anomaly for chiral particles

signature of "quantum atmosphere" of black hole

[S. B. Giddings, Physics Letters B 754, 39 (2016)]

The asymptotic Hawking radiation originates from a region outside of the black hole's horizon, its quantum atmosphere.



x/x_H distance in units of the horizon radius

Schwarzschild black hole $f(x) = 1 - x_H/x$

Quantum atmosphere is defined by the requirement of importance of quantum (anomalous) corrections: $T^2(x) \neq T^2_{\mathrm{TE}}(x)$

$$T_{\mathrm{TE}}(x) = \frac{T_0}{\sqrt{f(x)}}$$

Tolman-Ehrenfest temperature (classical)

Contribution from anomaly:

energy density $$\varepsilon = \frac{1}{2} C_w \left(\gamma T^2(x) + \varepsilon_q^{(1)} \right)$$

pressure $$p = \frac{1}{2} C_w \left(\gamma T^2(x) - \varepsilon_q^{(1)} \right)$$

energy flux $$J_\varepsilon = v_F^2 \Pi = C_g \frac{\pi}{12\hbar} k_B^2 T^2(x)$$

local temperature $$k_B^2 T^2(x) = \frac{k_B^2 T_H^2}{f} + \frac{6\hbar c}{\pi} \varepsilon_q^{(2)}$$

---

normalization: $J_H = (\pi/12\hbar) k_B^2 T_H^2$

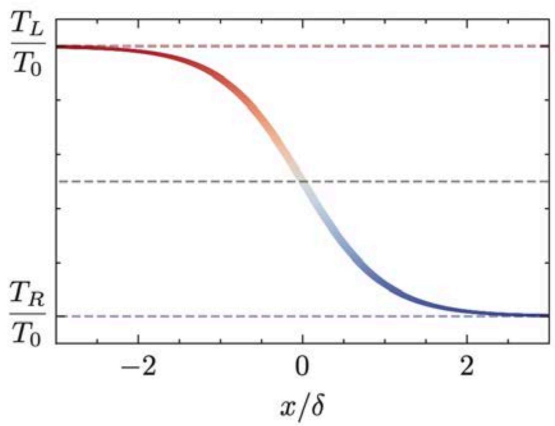
$\varepsilon_H = \frac{1}{2}\gamma T_H^2$

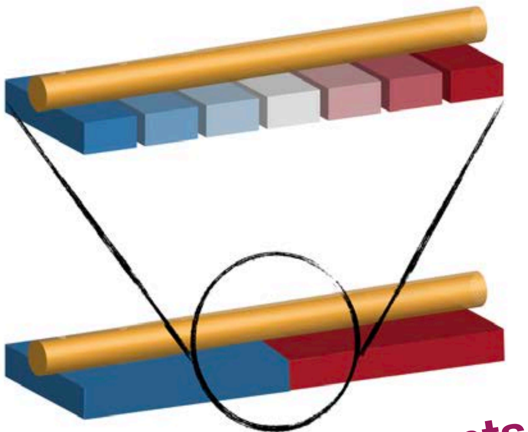
Measurable consequences

Heat-quench dynamics and Floquet heating states → ArXiv:2206.08784

This talk: Out-of-equilibrium states at inhomogeneous temperature

externally imposed profile $T(x)$





heating elements

Imposing externally a temperature profile $T(x)$:
Modified Gibbs measure

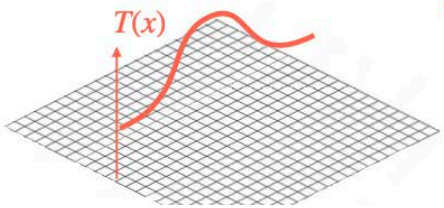
$$\rho = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}$$

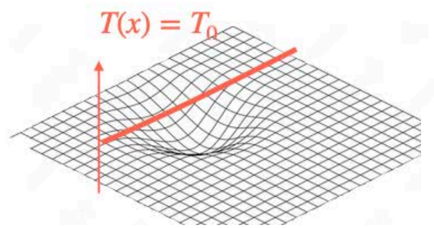
$$\text{with } \beta H = \int dx \frac{1}{k_B T(x)} h(x) = \frac{1}{k_B T_0} \int dx \frac{T_0}{T(x)} h(x)$$

Inhomogeneous $T(x)$ — **Flat** Space $\Leftrightarrow$ **Homogeneous** T_0 — **Curved** space

$$g_{\mu\nu} = \begin{pmatrix} \left(\frac{T_0}{T(x)}\right)^2 & 0 \\ 0 & -1 \end{pmatrix}$$

Extended Luttinger equivalence





(slide by Baptiste Bermond)

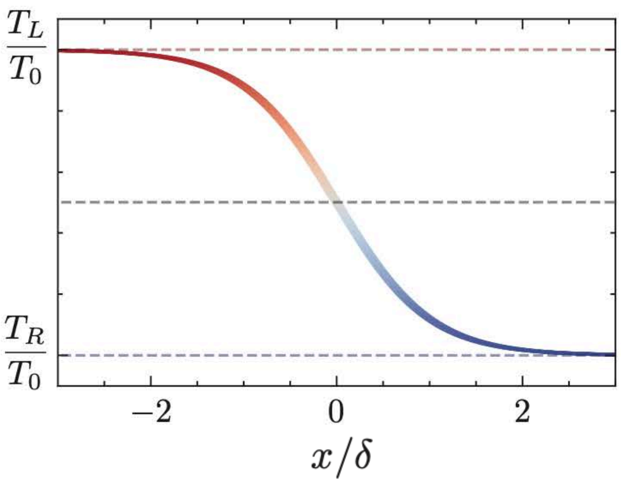
Effect of anomaly: two new temperature scales

$$l_1 = \left| \frac{T}{\partial_x T} \right| \quad \text{and} \quad l_2 = \sqrt{\left| \frac{T}{\partial_x^2 T} \right|}$$

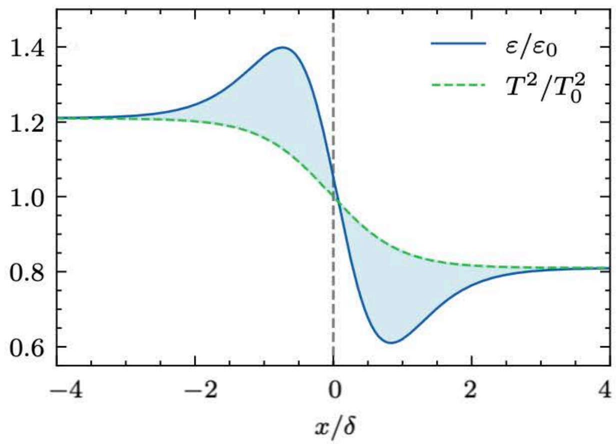
compare with
the thermal length

$$\lambda_T = \frac{\hbar v_F}{2\pi k_B T}$$

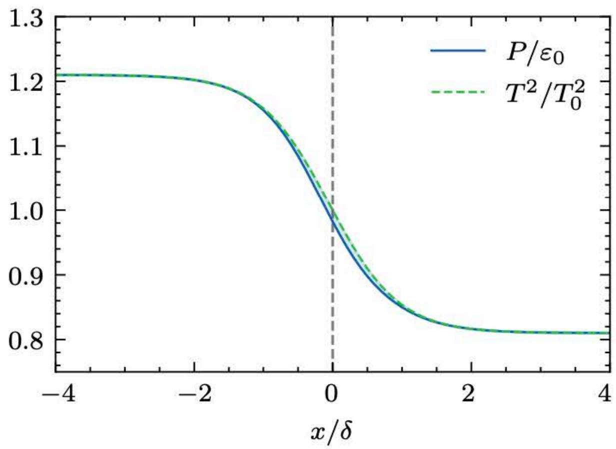
externally imposed temperature



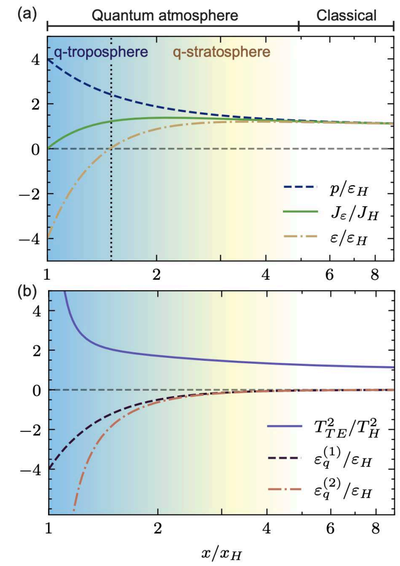
energy density



pressure



If we identify the local temperature from a measurement of the energy density or from a measurement of the pressure, we will get temperatures different from the externally imposed "classical" temperature.



Conclusions

Gravitational anomaly in (1+1) dimensional systems:

1) **modifies the Tolman-Ehrenfest and Luttinger relations between temperature and gravitational potential (and goes beyond the linear response)**

2) **leads to appearance of "gravitational zero modes": with constant temperature across inhomogeneous gravitational fields**

3) **generate two new length scales (seen in the quantum atmosphere of a black hole)**

4) **has potentially observable consequences for quantum wires (spin-Hall edge states, chiral edges of Quantum Hall systems) in stationary non-equilibrium**

5) **+ consequences for heat quenches and heat Floquet systems**