On the Nonrelativistic Expansion of GR

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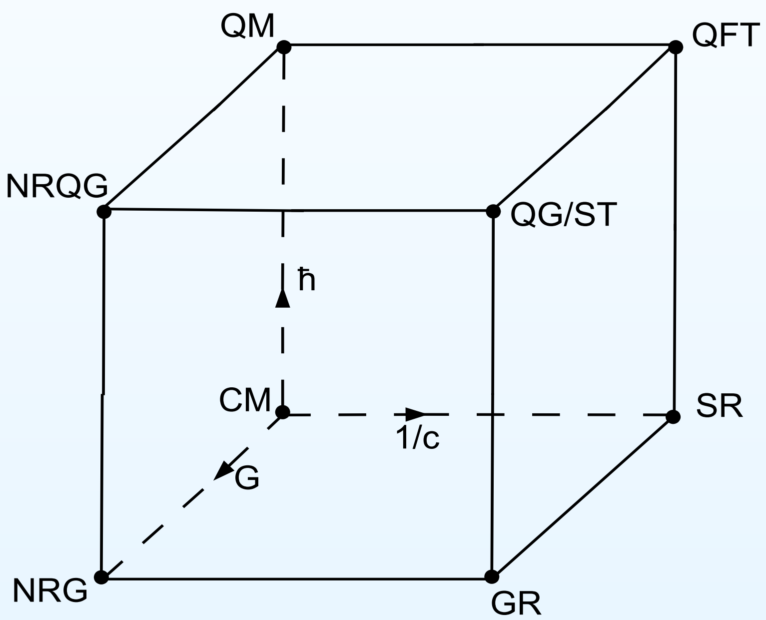
Introduction

- Bronstein cube

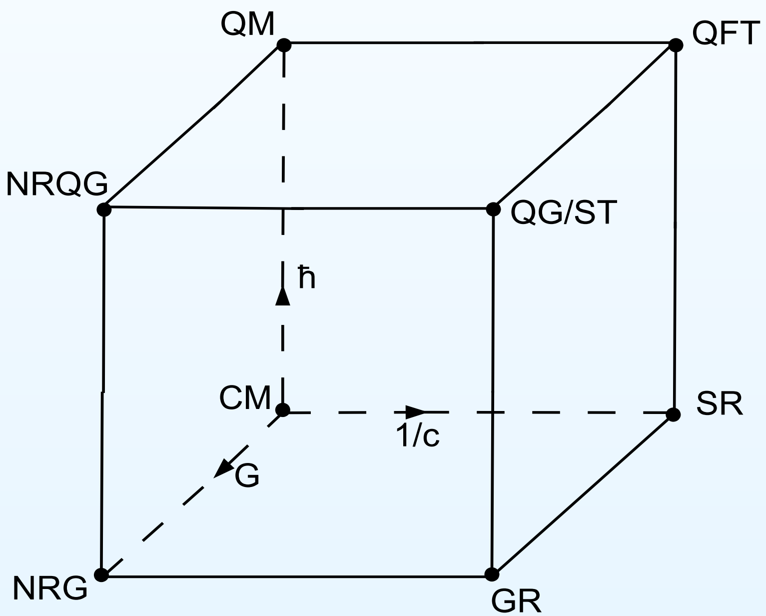


- NRG = non-relativistic gravity $\supset$ Newtonian gravity
- $1/c$ expansion $\supset$ post-Newtonian expansion
- NRQG = non-relativistic quantum gravity

- NRQG is a bit of a misnomer. There is no dynamical gravity to quantise. Think of quantum matter (described by quantum mechanics) backreacting with a background that reacts instantaneously.

Introduction

- Is there a well-defined non-relativistic limit of quantum gravity/string theory?



- In string theory there exist dualities between QG and specific QFTs (holography, AdS/CFT, ...).
- Do such dualities exist in the NR domain, i.e. can NR strings be dual to some NR field theory?

- Limits of AdS/CFT called Spin Matrix Theory give rise to NR strings dual to quantum mechanical limits of AdS/CFT. See e.g. [Harmark, Kristjansson, Orselli, 2007/8], [Harmark, JH, Obers, 2017].

Introduction: NR geometry

- Many of the recent developments in NR gravity, and NR string theory rely on an improved understanding of NR geometry.
- The most common example of a NR geometry is Newton–Cartan geometry which is the arena of every day life.
- Many other NR geometries have been found: type II Newton–Cartan, string Newton–Cartan, Aristotelian, Carrollian geometries ...
- Outside of GR and string theory, NR geometry has found applications in fluid dynamics, condensed matter physics, Hořava–Lifshitz gravity, 2D/3D gravity, ...

Introduction: NR approximations of GR

- ‘Drawbacks’ of Post-Newtonian approximation methods (Blanchet–Damour and Will–Wiseman approaches):
  - harmonic gauge
  - strong no-incoming radiation boundary condition
  - compactly supported matter

- An approach to address the first two issue is currently WIP [JH, Musaeus].

- We do not have a universal method to define non-relativistic approximations of GR coupled to any matter system (compact or non-compact) in a any gauge.

- The PN approximation is a weak field approximation. There is however also strong non-relativistic gravity. Can this be useful?

Introduction: PN corrections to quantum mechanics

- A very special case of NRQG is to take $G \to 0$ and study QM on a fixed background.
- Is there a coupling prescription for this?
- Suppose we know the $1/c$ corrections from SR how do we couple the system to geometries that are obtained from $1/c$ expansions of solutions of GR?
- What is the dynamics of a hydrogen atom in a Kerr background to some order in $1/c$?

Outline

- Newton–Cartan geometry
- $1/c$ expansion of GR
- QM on NR geometries

Newton–Cartan Geometry

metric : $\tau(a,b) = |t' - t| = \int_a^b \tau\,, \qquad \rho(a,b) = \|\vec{y} - \vec{x}\| = \int_a^b ds$

- Here $\tau = dt$ and $ds^2\big|_{t=\text{cst}} = \delta_{ij}dx^i dx^j$
- Write $\tau = \tau_\mu dx^\mu$ with coordinates $x^\mu = (t, x^i)$. Remove the restriction to $t = \text{cst}$ in ds^2 and write $ds^2 = h_{\mu\nu}dx^\mu dx^\nu$ as a quadratic form with signature $(0, 1, \ldots, 1)$.
- Under a Galilean boost with parameter $\lambda_\mu = (0, \vec{v})$

$$h_{\mu\nu} \to h_{\mu\nu} + \lambda_\mu \tau_\nu + \lambda_\nu \tau_\mu + \lambda^2 \tau_\mu \tau_\nu$$

- A manifold's tangent space is the flat version of the manifold. In general τ_μ, $h_{\mu\nu}$ and the parameter λ_μ are tensor fields.

τ	$=$	dt	absolute time
τ	$=$	$N dt$	absolute foliation

Newton–Cartan Geometry

- Mass is like electric charge $\Rightarrow$ gauge connection m_μ
- Fields on curved NC geometry

$$\delta S[\tau_\mu, h_{\mu\nu}, m_\mu] = \int d^4x e \left[\mathcal{E}^\mu \delta\tau_\mu + \frac{1}{2}\mathcal{T}^{\mu\nu}\delta h_{\mu\nu} + J^\mu \delta m_\mu \right]$$

$\mathcal{E}^\mu$	energy current
$\mathcal{T}^{\mu\nu}$	momentum-stress tensor
J^μ	mass current

- momentum = mass flux: $\mathcal{T}^{0i} = J^i \Leftrightarrow \delta m_\mu = \lambda_\mu$ and $\delta h_{\mu\nu} = \lambda_\mu \tau_\nu + \lambda_\nu \tau_\mu$
- mass conservation: $\partial_\mu(eJ^\mu) = 0 \Leftrightarrow \delta m_\mu = \partial_\mu \sigma$
- Triplet $(\tau_\mu, h_{\mu\nu}, m_\mu)$ with λ_μ, σ gauge redundancy defines a NC geometry.

Newton–Cartan Geometry

- Geodesic in NC geometry: Newton's equation

$$S = m \int d\lambda \left(\frac{h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{2\tau_\rho \dot{x}^\rho} - m_\mu \dot{x}^\mu \right)$$

- The time component of m_μ is Newton's potential.
- The fact that the mass is only an overall coupling is a manifestation of the equivalence principle.
- Schrödinger wavefunction on NC geometry

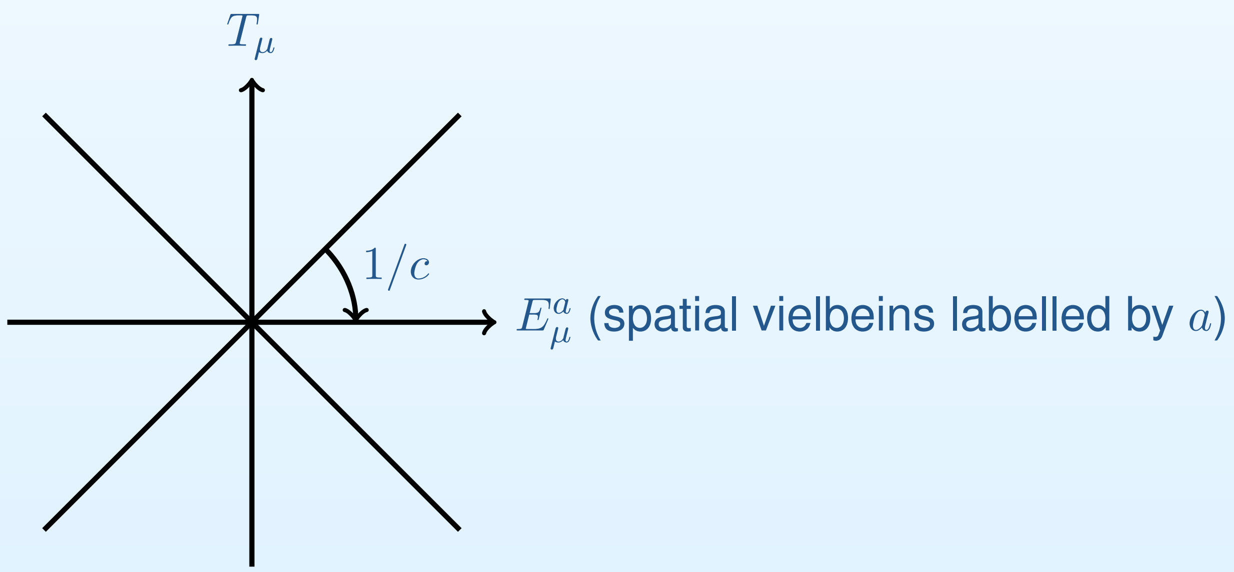
$$S = \int d^4 x e \left(im\psi^* v^\mu D_\mu \psi - im\psi v^\mu D_\mu \psi^* - h^{\mu\nu} D_\mu \psi D_\nu \psi^* \right)$$

  v^μ, $h^{\mu\nu}$ inverses of τ_μ and $h_{\mu\nu}$.
- $U(1)$ symmetry is gauged by m_μ: $D_\mu \psi = \partial_\mu \psi + imm_\mu \psi$.
- Mass conservation = conservation of probability.

$1/c$ expansion of GR

Review article: [JH, Obers, Oling, 2022]

- A convenient way to make the c-dependence of GR manifest is to write $g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$ and $g^{\mu\nu} = -\frac{1}{c^2} T^\mu T^\nu + \Pi^{\mu\nu}$.
- Signature of $\Pi_{\mu\nu}$ is $(0, 1, \ldots, 1)$.
- Light cones in tangent space have slope $1/c$:



$1/c$ expansion of GR

- So far we just reformulated GR in different variables. We will now assume that we can Taylor expand T_μ and $\Pi_{\mu\nu}$ in $1/c$:

$$T_\mu = \tau_\mu + \frac{1}{c^2} m_\mu + \frac{1}{c^4} B_\mu + \mathcal{O}(c^{-6}) \,, \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \frac{1}{c^2} \Phi_{\mu\nu} + \mathcal{O}(c^{-4})$$

- This is what leads to the covariant $1/c$ expansion.
- Note here only even powers. For odd powers see [Ergen, Hamamci, Van den Bleeken, 2020] and later in the PN expansion.
- This leads to the metric expansion:

$$g_{\mu\nu} = -c^2 \tau_\mu \tau_\nu + h_{\mu\nu} - 2\tau_{(\mu} m_{\nu)} + c^{-2} \left(\Phi_{\mu\nu} - m_\mu m_\nu - 2\tau_{(\mu} B_{\nu)} \right) + \mathcal{O}(c^{-4})$$

- The $1/c$ expansion of the metric was pioneered by [Dautcourt, 1990/97] and generalised in [Van den Bleeken, 2017], [JH, Hansen, Obers, 2018-20].

$1/c$ expansion of GR

- We can view the $1/c$ expansion as an expansion around a geometry described by τ_μ and $h_{\mu\nu}$ where all the higher order fields m_μ and $\Phi_{\mu\nu}$ are like gauge connections.
- Expanding the generator of infinitesimal diffeos: $\Xi^\mu = \xi^\mu + \frac{1}{c^2}\zeta^\mu + O(c^{-4})$ leads to gauge transformations for the subleading fields m_μ and $\Phi_{\mu\nu}$ w.r.t. subleading diffeos ζ^μ.
- Local Lorentz transformations acting on T_μ and $\Pi_{\mu\nu}$ also get expanded and lead to local Galilean transformations.
- Expanding the Einstein equations coupled to a point particle leads to Newtonian gravity:

$$\bar{R}_{\mu\nu} = 8\pi G \frac{d-2}{d-1}\rho\tau_\mu\tau_\nu\,, \qquad d\tau = 0$$

where we used the leading order Levi-Civita connection $\bar{\Gamma}^\rho_{\mu\nu}$.

$1/c$ expansion of GR: Examples of weak limits

$$ds^2_{\text{Schwarzschild}} = -c^2\left(1-\frac{2Gm}{c^2 r}\right)dt^2 + \left(1-\frac{2Gm}{c^2 r}\right)^{-1}dr^2 + r^2 d\Omega^2_{S^2}$$

$$ds^2_{\text{AdS(+)/dS(-)}} = -c^2\left(1\pm\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{1\pm\frac{r^2}{l^2}} + r^2 d\Omega^2_{S^2}$$

- Consider m independent of c.

$$\tau_\mu dx^\mu = dt\,,\qquad h_{\mu\nu}dx^\mu dx^\nu = dr^2 + r^2 d\Omega^2_{S^2}\,,\qquad m_\mu dx^\mu = -\frac{Gm}{r}dt$$

Point mass in flat spacetime with Newtonian pot. $\Phi = -\frac{Gm}{r}$.

- Take $l = c/H$ with the Hubble constant H independent of c.

$$\tau_\mu dx^\mu = dt\,,\qquad h_{\mu\nu}dx^\mu dx^\nu = dr^2 + r^2 d\Omega^2_{S^2}\,,\qquad m_\mu dx^\mu = \pm\frac{1}{2}H^2 r^2 dt$$

$1/c$ expansion of GR: Example of a strong limit

- Strong limit: $m = c^2 M$; M independent of c^2 [Van den Bleeken, 2017].

$$\tau_\mu dx^\mu = \sqrt{1 - \frac{2GM}{r}} dt \,, \qquad h_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

$$m_\mu dx^\mu = 0 = \Phi_{\mu\nu} dx^\mu dx^\nu$$

- This strong gravity expansion of the Schwarzschild metric is not captured by Newtonian gravity, but is still described as a Newton–Cartan geometry.

- This provides us with a different approximation of GR as compared to the post-Newtonian expansion.

- τ is no longer exact but $\tau \wedge d\tau = 0$ (hypersurface orthogonality). Strong limit captures gravitational time dilation: clocks tick slower/faster depending on position on a constant time slice.

QM on NR geometries

[JH, Have, Obers, Pikovski, to appear]

- Goal: given the Schrödinger equation for a quantum system in flat space time, including $1/c^2$ corrections, find a coupling prescription to couple this to $1/c^2$ expanded geometries.
- A top-down approach for the Schwarzschild metric was used in [Lämmerzahl, 1995] and generalised in [Schwartz, Giulini, 2018].
- Here the focus is on coupling prescriptions (i.e. general backgrounds and systematising results).
- We ignore backreaction: classically this leads to the Schrödinger–Newton equation which violates the superposition principle.

QM on NR geometries: The main idea

- KG equation and inner product on solution space:

$$-\frac{1}{c^2}\partial_t^2\phi + \nabla^2\phi - m^2c^2\phi = 0$$

$$\langle\phi_2|\phi_1\rangle = -\frac{i}{c^2}\int_{t=\text{cst}} d^3x\,(\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1)$$

- Define $\phi = \frac{1}{\sqrt{2m}}e^{-imc^2t}\psi$ with $\psi = \psi_{\text{LO}} + c^{-2}\psi_{\text{NLO}} + \cdots$

$$-\frac{1}{c^2}\partial_t^2\psi + 2im\partial_t\psi + \nabla^2\psi = 0$$

$$\langle\phi_2|\phi_1\rangle = \int_{t=\text{cst}} d^3x\left[\psi_2^*\psi_1 - \frac{i}{2mc^2}(\psi_1\partial_t\psi_2^* - \psi_2^*\partial_t\psi_1)\right]$$

- Define $\Psi = \psi - \frac{1}{4m^2c^2}\nabla^2\psi + \cdots$ s.t. $\langle\phi_2|\phi_1\rangle = \int_{t=\text{cst}} d^3x\Psi_2^*\Psi_1$

$$i\partial_t\Psi = -\frac{1}{2m}\nabla^2\Psi - \frac{1}{8m^3c^2}\nabla^4\Psi + O(c^{-4})$$

QM on NR geometries: coupling prescription

- We assume for simplicity flat NC spacetime: $\tau = dt$, $h = dx^i dx^i$.
- Again $\phi = \frac{1}{\sqrt{2m}} e^{-imc^2 t}\psi$ where $\psi = \psi_{\text{LO}} + c^{-2}\psi_{\text{NLO}} + \cdots$
- Using $\delta\phi = \Xi^\mu \partial_\mu \phi$ with $\Xi^\mu = \xi^\mu + c^{-2}\zeta^\mu + c^{-4}\chi^\mu + \cdots$ we find

$$\begin{aligned} \delta\psi_{\text{LO}} &= -im\zeta^t \psi_{\text{LO}} \\ \delta\psi_{\text{NLO}} &= -im\zeta^t \psi_{\text{NLO}} + \zeta^t \partial_t \psi_{\text{LO}} - im\chi^t \psi_{\text{LO}} + \zeta^i \partial_i \psi_{\text{LO}} \end{aligned}$$

- The ζ^t, χ^t are gauge transformation parameters with gauge fields m_μ, B_μ, that appear in the expansion of the vielbeins.
- We will deal with ζ^i separately.

QM on NR geometries: coupling prescription

- The coupling prescription is a statement about how to couple

$$\begin{aligned} \text{EOM}_{\text{LO}} &= 2im\partial_t\psi_{\text{LO}} + \nabla^2\psi_{\text{LO}} = 0 \\ \text{EOM}_{\text{NLO}} &= 2im\partial_t\psi_{\text{NLO}} + \nabla^2\psi_{\text{NLO}} - \partial_t^2\psi_{\text{LO}} = 0 \end{aligned}$$

  to the gauge fields m_μ, B_μ.
- The guiding principle is to find covariant derivatives such that

$$\begin{aligned} \delta\mathcal{D}_\mu\psi_{\text{LO}} &= -im\zeta^t\mathcal{D}_\mu\psi_{\text{LO}} \\ \delta\mathcal{D}_\mu\psi_{\text{NLO}} &= \zeta^t\partial_t\mathcal{D}_\mu\psi_{\text{LO}} - im\zeta^t\mathcal{D}_\mu\psi_{\text{NLO}} - im\chi^t\mathcal{D}_\mu\psi_{\text{LO}} \end{aligned}$$

- This leads to

$$\begin{aligned} \mathcal{D}_\mu\psi_{\text{LO}} &= \partial_\mu\psi_{\text{LO}} + imm_\mu\psi_{\text{LO}} \\ \mathcal{D}_\mu\psi_{\text{NLO}} &= \partial_\mu\psi_{\text{NLO}} + imm_\mu\psi_{\text{NLO}} + imB_\mu\psi_{\text{LO}} - m_\mu\mathcal{D}_t\psi_{\text{LO}} \end{aligned}$$

QM on NR geometries

- Minimal coupling then leads to

$$\begin{aligned}\text{EOM}_{\text{LO}} &= 2im\mathcal{D}_t\psi_{\text{LO}} + \mathcal{D}_i\mathcal{D}_i\psi_{\text{LO}} = 0\\ \text{EOM}_{\text{NLO}} &= 2im\mathcal{D}_t\psi_{\text{NLO}} + \mathcal{D}_i\mathcal{D}_i\psi_{\text{NLO}} - \frac{1}{c^2}\mathcal{D}_t\mathcal{D}_t\psi_{\text{LO}} + \cdots = 0\end{aligned}$$

- Terms on the dots are fixed by demanding covariance under NLO diffeos (ζ^i) and residual diffeos (ξ^i) of the LO geometry.

- The last step is to redefine ψ_{NLO} to a suitable $\hat{\psi}_{\text{NLO}}$ in order that the Klein–Gordon inner product becomes the standard one:

$$\langle\varphi_{\text{KG}}|\psi_{\text{KG}}\rangle = \int_{t=\text{cst}} d^dx\Big(\psi_{\text{LO}}+c^{-2}\hat{\psi}_{\text{NLO}}+\cdots\Big)\Big(\varphi^\star_{\text{LO}}+c^{-2}\hat{\varphi}^\star_{\text{NLO}}+\cdots\Big)$$

QM on NR geometries: Kerr geometry

$$ds^2_{\text{Kerr}} = ds^2_{\text{flat}} + \frac{\Sigma r_s R}{\Delta(R^2+a^2)} dR^2 + \frac{r_s R}{\Sigma}\left(-c dt + a \sin^2\Theta \, d\phi\right)^2$$

$$\Delta = R^2 + a^2 - r_s R \,, \qquad \Sigma = R^2 + a^2 \cos^2\Theta$$

- The parameters are $r_s = \frac{2GM}{c^2}$ and $a = \frac{J}{cM}$ with mass M and angular momentum J independent of c.
- ds^2_{flat} is written in oblate spherical coordinates (R, Θ, ϕ).
- Expanding in $1/c^2$ and transforming to ordinary spherical coordinates (r, θ, ϕ) leads to the 'Lense–Thirring metric':

$$\begin{aligned} ds^2_{\text{Kerr}} &= -\left(1 - \frac{2GM}{rc^2} + \frac{2GJ^2}{Mr^3c^4} P_2(\cos\theta)\right) c^2 dt^2 + \left(1 + \frac{2GM}{rc^2}\right) dr^2 \\ &\quad + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 - \frac{4GJ}{rc^2} \sin^2\theta \, dt d\phi + \mathcal{O}(c^{-4}) \end{aligned}$$

QM on NR geometries: Kerr geometry

- Using the coupling prescription and defining the Hamiltonian H as $i\partial_t \Psi = H\Psi$ where $\Psi = \psi_{(0)} + c^{-2}\psi_{(2)} + \ldots$ we find

$$
\begin{aligned}
H &= \frac{p^2}{2m} - \frac{GmM}{r} - \frac{p^4}{8c^2m^3} + \frac{GM}{c^2m}\left(-\frac{3}{2r^3}x^i p_i x^j p_j + \frac{1}{2r^3}L^2\right) \\
&\quad - \frac{mG^2M^2}{2c^2r^2} + \frac{2GJ}{c^2r^3}L_z + \frac{mGJ^2P_2(\cos\theta)}{Mc^2r^3} + \frac{GM}{4mc^2}\Delta(r^{-1})
\end{aligned}
$$

- This is the Hamiltonian of a spinless particle in a Kerr background up to order c^{-2}.
- This can be generalised to a spin 1/2 particle by starting with the Dirac equation.
- So far all top-down, but what if we want to study a hydrogen atom in a Kerr background?

Thank You!