'STAR-UBB Institute'
Online Series

Hyperfluids and Non-Riemannian Effects in Cosmology

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Apr 2023

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Outline

- Non-Riemannian Geometry: Conventions/Notation
- Hyperfluids, Torsion and Non-metricity in Cosmology
- Quadratic Metric-Affine Gravity
- The Cosmology of quadratic MAG
- Conclusions/Further Prospects

The talk is based on the papers

- "Cosmological Hyperfluids, Torsion and Non-metricity" Published in: Eur.Phys.J.C 80 (2020) 11, 1042 • e-Print: 2003.07384 [gr-qc] (Damianos Iosifidis)
- "The Perfect Hyperfluid of Metric-Affine Gravity: The Foundation" Published in: JCAP 04 (2021) 072 • e-Print: 2101.07289 [gr-qc] (Damianos Iosifidis)
- Cosmology of quadratic metric-affine gravity Published in: Phys.Rev.D 105 (2022) 2, 2 • e-Print: 2109.06167 [gr-qc] (Damianos Iosifidis and Lucrezia Ravera)

Metric-Affine Gravity

Metric Gravity

- $\Gamma^{\alpha}{}_{\mu\nu} \rightarrow$ *torsionless* , metric compatibility $\nabla_{\sigma} g_{\mu\nu} = 0$
- $S = S_{Gravity} + S_{Matter} = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Phi) \right]$

Teleparallel/Symmetric Teleparallel Gravity

- $R^{\alpha}{}_{\beta\mu\nu} = 0$, $\nabla_{\sigma} g_{\mu\nu} = 0$ but $S_{\mu\nu}{}^{\alpha} = \Gamma^{\alpha}{}_{[\mu\nu]} \neq 0$
- $R^{\alpha}{}_{\beta\mu\nu} = 0$, $S_{\mu\nu}{}^{\alpha} = 0$ but $Q_{\alpha\mu\nu} = -\nabla_{\alpha} g_{\mu\nu} \neq 0$

Metric-Affine Gravity (MAG)

- $S = \int d^n x \sqrt{-g} \left[\mathcal{L}_G(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^{\alpha}{}_{\mu\nu}, \Phi) \right] \Rightarrow$ No a priori constraints on the geometry.

Geometrical Objects

Two distinctively different notions on a manifold

- Metric Tensor $g_{\mu\nu}$: Defines distances,lengths and dot products

$$||\alpha||^2 := \alpha^\mu \alpha^\nu g_{\mu\nu} \;, \;\; (\alpha \cdot \beta) := \alpha^\mu \beta^\nu g_{\mu\nu}$$

- Affine-Connection $\Gamma^\lambda{}_{\mu\nu}$: Defines parallel transport of tensor fields on the manifold

$$\nabla_\lambda u^\mu = \partial_\lambda u^\mu + \Gamma^\mu{}_{\nu\lambda} u^\nu$$

The two need not be related a priori! Their relation may be found after solving the field equations!

Geometrical Objects

Torsion

- $\nabla_{[\mu}\nabla_{\nu]}\phi = S_{\mu\nu}{}^{\lambda}\nabla_{\lambda}\phi$, Torsion Tensor $S_{\mu\nu}{}^{\lambda} := \Gamma^{\lambda}{}_{[\mu\nu]}$

Curvature

- $[\nabla_{\alpha},\nabla_{\beta}]u^{\mu} = R^{\mu}{}_{\nu\alpha\beta}u^{\nu} + 2S_{\alpha\beta}{}^{\nu}\nabla_{\nu}u^{\mu}$
  Curvature Tensor: $R^{\mu}{}_{\nu\alpha\beta} := 2\partial_{[\alpha}\Gamma^{\mu}{}_{|\nu|\beta]} + 2\Gamma^{\mu}{}_{\rho[\alpha}\Gamma^{\rho}{}_{|\nu|\beta]}$

Non-Metricity

- $Q_{\alpha\mu\nu} := -\nabla_{\alpha}g_{\mu\nu} = -\partial_{\alpha}g_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\alpha}g_{\lambda\nu} + \Gamma^{\lambda}{}_{\nu\alpha}g_{\lambda\mu}$

Contractions

Contractions of Curvature

- Ricci Tensor: $R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}$
- Homothetic Curvature: $\widehat{R}_{\alpha\beta} := R^{\mu}{}_{\mu\alpha\beta}$
- 2nd Ricci Tensor: $\check{R}_{\mu\nu} := R_{\mu\alpha\beta\nu}g^{\alpha\beta}$
- Ricci Scalar: $R := R_{\mu\nu}g^{\mu\nu} = -\check{R}_{\mu\nu}g^{\mu\nu}$

Torsion/Non-metricity related vectors

$$S_{\mu} = S_{\mu\lambda}{}^{\lambda} \, , \qquad \tilde{S}^{\mu} = \epsilon^{\mu\nu\rho\sigma}S_{\nu\rho\sigma} \;\; (only \;\; for \;\; n=4)$$

$$Q_{\mu} = g^{\alpha\beta}Q_{\mu\alpha\beta} \, , \qquad \tilde{Q}_{\mu} = g^{\rho\alpha}Q_{\rho\alpha\mu}$$

Affine Connection

Affine connection decomposition

$$\Gamma^{\lambda}{}_{\mu\nu} = \tilde{\Gamma}^{\lambda}{}_{\mu\nu} + \frac{1}{2} g^{\alpha\lambda}(Q_{\mu\nu\alpha} + Q_{\nu\alpha\mu} - Q_{\alpha\mu\nu}) - g^{\alpha\lambda}(S_{\alpha\mu\nu} + S_{\alpha\nu\mu} - S_{\mu\nu\alpha})$$

where $\tilde{\Gamma}^{\lambda}{}_{\mu\nu} := \frac{1}{2} g^{\alpha\lambda}(\partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\alpha\mu} - \partial_{\alpha} g_{\mu\nu})$ is the Levi-Civita part of the connection. Distortion: $N^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \tilde{\Gamma}^{\lambda}{}_{\mu\nu}$

Decompositions

Each quantity $\Rightarrow$ decomposed into Riemannian and non-Riemannian counterparts. Example:

$$R = \tilde{R} + S_{\mu\nu\alpha} S^{\mu\nu\alpha} - 2 S_{\mu\nu\alpha} S^{\alpha\mu\nu} - 4 S_{\mu} S^{\mu} - 4 \tilde{\nabla}_{\mu} S^{\mu}$$
$$+ \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} - \frac{1}{4} Q_{\mu} Q^{\mu} + \frac{1}{2} Q_{\mu} \tilde{Q}^{\mu}$$
$$+ 2 Q_{\alpha\mu\nu} S^{\alpha\mu\nu} + 2 S_{\mu}(\tilde{Q}^{\mu} - Q^{\mu}) + \tilde{\nabla}_{\mu}(\tilde{Q}^{\mu} - Q^{\mu} - 4 S^{\mu})$$

Hypermomentum, Canonical and Metrical Energy Momentum Tensors

Metrical and Canonical Energy Momentum Tensor

Metrical: $T_{\alpha\beta} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}$. Canonical: $t^{\mu}{}_{c} = \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta e_{\mu}{}^{c}}$

Hypermomentum Tensor

Hypermomentum: $\Delta_{\lambda}{}^{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$

Relation Between Energy Tensors

$$t^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} - \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu})$$

where $\hat{\nabla}_{\nu} = 2S_{\nu} - \nabla_{\nu}$.

Conservation Laws

Working in exterior calculus from the GL and diff invariance we get

From GL

$$t^{\mu}{}_{\lambda} = T^{\mu}{}_{\lambda} - \frac{1}{2\sqrt{-g}}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu})$$

From Diff

$$\frac{1}{\sqrt{-g}}\hat{\nabla}_{\mu}(\sqrt{-g}t^{\mu}{}_{\alpha}) = -\frac{1}{2}\Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha} + \frac{1}{2}Q_{\alpha\mu\nu}T^{\mu\nu} + 2S_{\alpha\mu\nu}t^{\mu\nu}$$

From Diff using coordinates

$$\sqrt{-g}(2\tilde{\nabla}_{\mu}T^{\mu}{}_{\alpha} - \Delta^{\lambda\mu\nu}R_{\lambda\mu\nu\alpha}) + \hat{\nabla}_{\mu}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\alpha}{}^{\mu\nu})$$
$$+2S_{\mu\alpha}{}^{\lambda}\hat{\nabla}_{\nu}(\sqrt{-g}\Delta_{\lambda}{}^{\mu\nu}) = 0$$

Homogeneous Cosmology with Torsion and non-metricity

- Applying Cosmological Principle to Torsion [Tsamparlis,1979]:
  $S_{01}{}^{1} = S_{02}{}^{2} = S_{03}{}^{3} = ... = S_{0m}{}^{m} \neq 0 \quad (no\ sum)$
  $S_{ijk} \propto \epsilon_{ijk} \neq 0$ (only for $n = 4$)
- Applying it to Non-Metricity[Minkevich,1998]:
  $Q_{011} = ... = Q_{0mm} \neq 0,\ Q_{110} = ... = Q_{mm0} \neq 0,$
  $Q_{000} \neq 0$ Here $m = n - 1$=spatial space dim
  $\Longrightarrow$ The rest vanish!

Covariant Forms

The covariant forms of the above read [D.I,2020]

- $S^{(n)}_{\mu\nu\alpha} = 2u_{[\mu}h_{\nu]\alpha}\Phi(t) + \epsilon_{\mu\nu\alpha\rho}u^{\rho}P(t)\delta_{n,4}$
- $Q_{\alpha\mu\nu} = A(t)u_{\alpha}h_{\mu\nu} + B(t)h_{\alpha(\mu}u_{\nu)} + C(t)u_{\alpha}u_{\mu}u_{\nu}, \quad \forall$ n

$N^{(n)}_{\alpha\mu\nu} = X(t)u_{\alpha}h_{\mu\nu} + Y(t)u_{\mu}h_{\alpha\nu} + Z(t)u_{\nu}h_{\alpha\mu} + V(t)u_{\alpha}u_{\mu}u_{\nu} + \epsilon_{\alpha\mu\nu\lambda}u^{\lambda}W(t)\delta_{n,4}$ for the distortion.

Isotropic Hypermomentum

Imposing Cosm. Principle to Hypermomentum($\mathcal{L}_{\xi^i}\Delta_{\alpha\mu\nu}=0$)
$\Delta_{i00}=\Delta_{0i0}=\Delta_{00i}=0$,
$\Delta_{110}=...=\Delta_{mm0}$,$\Delta_{011}=...=\Delta_{0mm}$(no sum)

Covariant Form of Hypermomentum

Using an $1+(n-1)$ split we get the covariant form[D.I,2020]:

- $\Delta^{(n)}_{\alpha\mu\nu}=\phi h_{\mu\alpha}u_\nu+\chi h_{\nu\alpha}u_\mu+\psi u_\alpha h_{\mu\nu}+\omega u_\alpha u_\mu u_\nu+\delta_{n,4}\epsilon_{\alpha\mu\nu\kappa}u^\kappa\zeta$

Most General form of Hypermomentum respecting isotropy!

Comments

1. In an FLRW $\phi,\chi,...$ depend only on time t. If homogeneity is relaxed $\phi=\phi(t,x^i)$ etc. (more about it later)
2. Hypermomentum generally contributes 5 dof in a Cosmological setting($n=4$). (and 4 dof for $n\neq 4$).

Hypermomentum Decomposition (Matter with Microstructure)

- Spin Part: $\Delta_{[\alpha\mu]\nu} = (\psi - \chi)u_{[\alpha}h_{\mu]\nu} + \delta_{n,4}\epsilon_{\alpha\mu\nu\kappa}u^{\kappa}\zeta$
- Dilation Part: $\Delta_{\nu} := \Delta_{\alpha\mu\nu}g^{\alpha\mu} = \Big[(n-1)\phi - \omega\Big]u_{\nu}$
- Shear Part: $\breve{\Delta}_{\alpha\mu\nu} = \Delta_{(\alpha\mu)\nu} - \frac{1}{n}g_{\alpha\mu}\Delta_{\nu} = \frac{(\phi+\omega)}{n}\Big[h_{\alpha\mu} + (n-1)u_{\alpha}u_{\mu}\Big]u_{\nu} + (\psi + \chi)u_{(\mu}h_{\alpha)\nu}$

Sourcing Torsion and Non-Metricity ($5 = 2 + 3$)

By means of the connection field eqs, the above parts act as sources producing spacetime torsion and non-metricity (see example later).

The Perfect (Ideal) Hyperfluid [D.I, 2020]

Energy Momentum:

$T_{\mu\nu} = t_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}$

Hypermomentum :

$\Delta^{(n)}_{\alpha\mu\nu} = \phi h_{\mu\alpha} u_\nu + \chi h_{\nu\alpha} u_\mu + \psi u_\alpha h_{\mu\nu} + \omega u_\alpha u_\mu u_\nu + \delta_{n,4} \epsilon_{\alpha\mu\nu\kappa} u^\kappa \zeta$

Conservation laws (obtained from diff invariance)

$\tilde{\nabla}_\mu T^\mu{}_\nu = \frac{1}{2} \Delta^{\alpha\beta\gamma} R_{\alpha\beta\gamma\nu}. \quad \hat{\nabla}_\nu \left(\sqrt{-g} \Delta_\lambda{}^{\mu\nu} \right) = 0$

We call it hypermomentum preserving.

Note

The conservation law for hypermomentum (2nd eq. above) in an FLRW Universe really contains 2 independent eqs for the 5 fields.
$\Rightarrow$ 3 eqs of state must be provided.

Generalization: There exists a Perfect Hyperfluid, generalizing the Perfect Fluid notion of GR, for which: (D.I. 2021, JCAP)

$$t_{\mu\nu} = \tilde{\rho} u_\mu u_\nu + \tilde{p} h_{\mu\nu} \quad , \quad T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} \tag{1}$$

$$\Delta^{(n)}_{\alpha\mu\nu} = \phi h_{\mu\alpha} u_\nu + \chi h_{\nu\alpha} u_\mu + \psi u_\alpha h_{\mu\nu} + \omega u_\alpha u_\mu u_\nu + \delta_{n,4} \epsilon_{\alpha\mu\nu\kappa} u^\kappa \zeta \tag{2}$$

These sources are subject to the conservation laws:

$$\tilde{\nabla}_\mu t^\mu{}_\alpha = \frac{1}{2} \Delta^{\lambda\mu\nu} R_{\lambda\mu\nu\alpha} + \frac{1}{2} Q_{\alpha\mu\nu} (t^{\mu\nu} - T^{\mu\nu}) \tag{3}$$

$$t^\mu{}_\lambda = T^\mu{}_\lambda - \frac{1}{2\sqrt{-g}} \hat{\nabla}_\nu (\sqrt{-g} \Delta_\lambda{}^{\mu\nu}) \tag{4}$$

Remark

The Perfect Hyperfluid is a direct generalization of the Perfect Fluid description where now the microscopic characteristics of matter are also taken into account.

Parity Even Quadratic MAG Theory

Quadratic MAG

$$S[g,\Gamma,\Phi]=\frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[R+b_1 S_{\alpha\mu\nu}S^{\alpha\mu\nu}+b_2 S_{\alpha\mu\nu}S^{\mu\nu\alpha}$$
$$+b_3 S_\mu S^\mu+a_1 Q_{\alpha\mu\nu}Q^{\alpha\mu\nu}+a_2 Q_{\alpha\mu\nu}Q^{\mu\nu\alpha}+a_3 Q_\mu Q^\mu+a_4 q_\mu q^\mu$$
$$+a_5 Q_\mu q^\mu+c_1 Q_{\alpha\mu\nu}S^{\alpha\mu\nu}+c_2 Q_\mu S^\mu+c_3 q_\mu S^\mu\Big]+S_M[g,\Gamma,\Phi] \quad (5)$$

Matter

$S_M[g,\Gamma,\Phi]$ is a generic matter part that couples to the connection as well. Later this will be chosen to represent the Perfect Hyperfluid.

Field Equations

g-Variation

$$
\begin{aligned}
&R_{(\mu\nu)} - \frac{R}{2} g_{\mu\nu} - \frac{\mathcal{L}^{(2)}_{even}}{2} g_{\mu\nu} \\
&- \frac{1}{\sqrt{-g}} (\nabla_\alpha - 2S_\alpha) \Big[\sqrt{-g} \Big(c_1 S^\alpha{}_{(\mu\nu)} + c_2 g_{\mu\nu} S^\alpha + c_3 \delta^\alpha_{(\mu} S_{\nu)} \Big) \Big] \\
&- \frac{1}{\sqrt{-g}} (\nabla_\alpha - 2S_\alpha) \Big[\sqrt{-g} \Big(2a_1 Q^\alpha{}_{\mu\nu} + 2a_2 Q_{(\mu\nu)}{}^\alpha \\
&+ (2a_3 Q^\alpha + a_5 q^\alpha) g_{\mu\nu} + (2a_4 q_{(\mu} + a_5 Q_{(\mu}) \delta^\alpha_{\nu)} \Big) \Big] \\
&+ a_1 (Q_{\mu\alpha\beta} Q_\nu{}^{\alpha\beta} - 2 Q_{\alpha\beta\mu} Q^{\alpha\beta}{}_\nu) - a_2 Q_{\alpha\beta(\mu} Q^{\beta\alpha}{}_{\nu)} + a_3 (Q_\mu Q_\nu - 2 Q^\alpha Q_{\alpha\mu\nu}) \\
&+ b_1 (2 S_{\nu\alpha\beta} S_\mu{}^{\alpha\beta} - S_{\alpha\beta\mu} S^{\alpha\beta}{}_\nu) - b_2 S_{(\mu}{}^{\beta\alpha} S_{\nu)\alpha\beta} + b_3 S_\mu S_\nu \\
&+ c_1 (Q_{(\mu}{}^{\alpha\beta} S_{\nu)\alpha\beta} - S_{\alpha\beta(\mu} Q^{\alpha\beta}{}_{\nu)}) + c_2 (S_{(\mu} Q_{\nu)} - S^\alpha Q_{\alpha\mu\nu}) = \kappa T_{\mu\nu}
\end{aligned}
\tag{6}
$$

Γ-Variation

$$\left(\frac{Q_\lambda}{2}+2S_\lambda\right)g^{\mu\nu}-Q_\lambda{}^{\mu\nu}-2S_\lambda{}^{\mu\nu}+\left(q^\mu-\frac{Q^\mu}{2}-2S^\mu\right)\delta^\nu_\lambda$$
$$+4a_1Q^{\nu\mu}{}_\lambda+2a_2(Q^{\mu\nu}{}_\lambda+Q_\lambda{}^{\mu\nu})+2b_1S^{\mu\nu}{}_\lambda+2b_2S_\lambda{}^{[\mu\nu]}$$
$$+c_1\left(S^{\nu\mu}{}_\lambda-S_\lambda{}^{\nu\mu}+Q^{[\mu\nu]}{}_\lambda\right)+\delta^\mu_\lambda\left(4a_3Q^\nu+2a_5q^\nu+2c_2S^\nu\right)$$
$$+\delta^\nu_\lambda\left(a_5Q^\mu+2a_4q^\mu+c_3S^\mu\right)+g^{\mu\nu}\left(a_5Q_\lambda+2a_4q_\lambda+c_3S_\lambda\right)$$
$$+\left(c_2Q^{[\mu}+c_3q^{[\mu}+2b_3S^{[\mu}\right)\delta^{\nu]}_\lambda=\kappa\Delta_\lambda{}^{\mu\nu} \tag{7}$$

Connecting them to their sources

Using the connection field eqs we can express the torsion and non-metricity functions in terms of their sources (hypermomentum components)

$$\begin{aligned} A &= \kappa\left(-\lambda_{11}\omega + \lambda_{12}\psi + \lambda_{13}\phi + \lambda_{14}\chi\right) , \\ B &= \kappa\left(-\lambda_{21}\omega + \lambda_{22}\psi + \lambda_{23}\phi + \lambda_{24}\chi\right) , \\ C &= \kappa\left(-\lambda_{31}\omega + \lambda_{32}\psi + \lambda_{33}\phi + \lambda_{34}\chi\right) , \\ \Phi &= \kappa\left(-\lambda_{41}\omega + \lambda_{42}\psi + \lambda_{43}\phi + \lambda_{44}\chi\right) , \end{aligned} \tag{8}$$

and $P = \kappa\lambda_{00}\zeta$, where the λ's depend on the a's, b's, and c's.

Then, using a post-Riemannian expansion of the metric field equations and considering an FLRW background the modified Friedmann equations can be obtained. These look rather lengthy in general (see Phys. Rev. D 105, 024007).

Note

Due to the high symmetry of the FLRW spscetime "only" 8 out of the 11 quadratic invariants are independent.

As a representative member we consider the subsector

$$S[g,\Gamma,\varphi]=\frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[R+b_3 S_\mu S^\mu + a_3 Q_\mu Q^\mu\Big]+S_{\text{hyp}}\,, \quad (9)$$

that is we set $b_1=a_1=a_2=a_4=a_5=c_1=c_2=c_3=0$ in the general quadratic model.
From the metric field equations, we then obtain the modified Friedmann eqns in the presence of a Perfect Hyperfluid.

Modified Friedmann Equations

The acceleration equation is found to be

$$\frac{\ddot{a}}{a}=-\frac{\kappa}{6}(\rho+3p)+\frac{a_3}{3}\Big(3A-C\Big)^2+3b_3\Phi^2-\frac{1}{3}\Big(\dot{I}+HI\Big)+H\Big(2\Phi+A+\frac{C}{2}\Big)$$

$-\Big(2\Phi+\frac{A}{2}\Big)(A+C)+2\dot{\Phi}+\frac{\dot{A}}{2}$, (10) where $I=2a_3(3A-C)$.

Variant of the 1st Friedmann eqn reads

$$6\left[\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right]-3\Big(-\frac{1}{2}+3a_3\Big)A^2-a_3C^2-3\Big(3b_3-8\Big)\Phi^2$$
$$-6P^2-\frac{3}{4}AB+6a_3AC+\frac{3}{4}BC$$
$$+12A\Phi-6B\Phi=-\Big(\dot{f}+3Hf\Big)-\kappa\Big(-\rho+3p\Big), \qquad (11)$$

where $f=\frac{3}{2}B-8a_3C+3(8a_3-1)A-12\Phi$

Pure Dilation (and hypermomentum preserving) Case

This implies

$$\zeta = 0\,, \quad \psi = 0\,, \quad \chi = 0\,, \quad \omega = -\phi\,, \tag{12}$$

Therefore, we are left with one non-vanishing source variable.

$$\begin{aligned}
P &= 0\,,\\
A &= \kappa\left[\frac{(16-3b_3)}{8a_3(32-3b_3)+9b_3}\right]\phi\,,\\
B &= \kappa\left[\frac{6b_3}{8a_3(32-3b_3)+9b_3}\right]\phi\,,\\
C &= -\kappa\left[\frac{(16+3b_3)}{8a_3(32-3b_3)+9b_3}\right]\phi\,,\\
\Phi &= -\kappa\left[\frac{4}{8a_3(32-3b_3)+9b_3}\right]\phi\,.
\end{aligned} \tag{13}$$

Friedmann Equation(After employing the CL's)

$$H^2 = -\frac{\left[3072 + 8a_3\left(32 - 3b_3\right)^2 - 288b_3 - 81b_3^2\right]}{12\left[9b_3 - 8a_3\left(3b_3 - 32\right)\right]^2}\kappa^2\phi^2 + \frac{\kappa}{3}\rho\,. \tag{14}$$

Conservation Laws

The conservation laws in this case reduce to:

$$\dot{\phi} + 3H\phi = 0\,, \tag{15}$$

$$\dot{\rho} + 3H(1+w)\rho = 0\,. \tag{16}$$

Note that in this case the density decouples and evolves as in the usual perfect fluid continuity equation, while also for ϕ we observe an analogous evolution which, in particular, mimics that of dust.

Final System

$$H^2 = \frac{\kappa}{3}\rho + \mathcal{B}\frac{\kappa^2\phi^2}{4} \tag{17}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(1+3w)\rho - \mathcal{B}\frac{\kappa^2\phi^2}{2} \tag{18}$$

$$\dot{\rho} + 3H(1+w)\rho = 0 \tag{19}$$

$$\dot{\phi} + 3H\phi = 0 \tag{20}$$

with

$$\mathcal{B} = -\frac{(32-3b_3)^2\left[-1024+3b_3(64+3b_3)\right]}{9\left[2048-3b_3(128+9b_3)\right]^2}\,,$$

Note that in this context pure dilation hypermomentum effectively acts as stiff matter! For early times $a(t) \propto t^{1/3}$.

Pure Spin Case

In this case we need to impose

$$\phi = 0\,, \quad \omega = 0\,, \quad \chi = -\psi\,, \tag{21}$$

and as a result the non-metricity and torsion cosmological functions become

$$P = -\frac{\kappa}{2}\zeta\,, \tag{22}$$

$$A = -\frac{3b_3}{b_0}(3 + 8a_3)\kappa\psi\,, \tag{23}$$

$$B = \frac{96a_3b_3}{b_0}\kappa\psi\,, \tag{24}$$

$$C = \frac{b_3}{b_0}9\left(1 - 8a_3\right)\kappa\psi\,, \tag{25}$$

$$\Phi = -\frac{64a_3}{b_0}\kappa\psi\,, \;\; b_0 := \frac{1}{8a_3(3b_3 - 32) - 9b_3}\,. \tag{26}$$

Conservation laws

From the conservation law (4) we get the two relations,

$$p_c - p = \frac{1}{4} B\psi \,, \tag{28}$$

$$\rho_c - \rho = \frac{3}{4} B\psi \,, \tag{29}$$

which also imply that[a]

$$\rho_c - \rho = 3(p_c - p) \,. \tag{30}$$

[a] Note that this seems to be a "radiation-like" equation of state $\hat{p} = \hat{\rho}/3$ for the net pressure $\hat{p} = (p_c - p)$ and density $\hat{\rho} = (\rho_c - \rho)$.

We notice, from the above equations, that it is quite crucial in this case to consider a generalized non-preserving hypermomentum, namely $\rho_c \neq \rho$ and $p_c \neq p$ need to hold true.

Using also the connection field eqns we obtain the net density and pressure

$$\rho_c = \rho + \frac{72 a_3 b_3}{b_0}\kappa\psi^2 \tag{31}$$

$$p_c = p + \frac{24 a_3 b_3}{b_0}\kappa\psi^2 \,. \tag{32}$$

respectively. On the other hand, the modified continuity equation boils down to

$$\dot{\rho}_c + 3H(\rho_c + p_c) = -\frac{3}{2}\mu_1\kappa\psi\Big(\dot{\psi} + H\psi\Big) - 3\psi\frac{\ddot{a}}{a}\,, \tag{33}$$

where

$$\mu_1 := \frac{28 a_3 + 48 a_3 b_3 + \frac{9}{2} b_3}{b_0}\,.$$

Hypermomentum domination

Now, during a hypermomentum dominated era (very early Universe), the main contributions in (31) and (32) would be the ones $\propto \psi^2$. In other words, the classical perfect fluid contributions ρ and p can be ignored. Then,

$$\frac{\ddot{a}}{a} = \frac{\kappa}{2b_0^2}\left(\dot{\psi} + H\psi - 2304\kappa a_3^2 b_3^2 \psi^2\right) . \tag{34}$$

We may then eliminate the double derivative term $\frac{\ddot{a}}{a}$ from the continuity equation (also ignoring ρ, p), which results in

$$\dot{\psi} + (1+\mu_2)\, H\psi + \mu_3 \kappa \psi^2 = 0 \,, \tag{35}$$

where $\mu_2 := 96 a_3 b_3 \nu_1 \, , \mu_3 := -\frac{2304}{b_0} a_3^2 b_3^2 \nu_1$
$\nu_1 := \frac{1}{[-9b_3 + 8a_3(-32+15b_3) + b_0\mu_1]}$

Friedmann Equation

$$H^2 = \frac{\kappa}{2b_0^2}\left[\pi_1\dot{\psi} + (2+\pi_1)H\psi + \frac{\kappa}{2}\pi_2\psi^2\right] + \frac{\kappa^2}{4}\zeta^2\,, \qquad (36)$$

where

$$\pi_1 := 1 + 9b_0 b_3 - 8a_3 b_0(-32+3b_3)\,,$$
$$\pi_2 := -81b_3^2 - 144a_3 b_3(-32+3b_3) + 64a_3^2(-1024+192b_3+99b_3^2)\,.$$

Hyperfluid Equation of state

It is also natural to assume an equation of state $\zeta = w_\zeta \psi$ among the spin variables.

Solutions

$$H^2 - \lambda_1 H\psi - \lambda_2 \psi^2 = 0\,, \tag{37}$$

where

$$\lambda_1 := \frac{\kappa}{2b_0^2}\Big(2 - \mu_2\pi_1\Big)\,, \quad \lambda_2 := \frac{\kappa^2}{2b_0^2}\left(-\pi_1\mu_3 + \frac{\pi_2 + b_0^2 w_\zeta^2}{2}\right)\,. \tag{38}$$

Then, we observe that (37) is a simple quadratic equation, which, considering H as the unknown variable, admits the solutions (for $\lambda_1^2 + 4\lambda_2 > 0$)

$$H = \lambda_0\psi\,, \quad \lambda_0 = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 + 4\lambda_2}}{2}\,. \tag{39}$$

Substituting this back into the continuity equation for ψ it follows that $\longrightarrow$

Solutions

$$\dot{\psi} = -\mu_0 \psi^2 \,, \quad \mu_0 = \lambda_0 (1 + \mu_2) + \kappa \mu_3 \,, \tag{40}$$

which trivially integrates to

$$\psi(t) = \frac{1}{c_1 + \mu_0 t} \,, \tag{41}$$

where c_1 is an arbitrary integration constant. Finally, substituting this form for ψ back into (39) and integrating, we find the following expression for the scale factor:

$$a(t) = c_2 (c_1 + \mu_0 t)^{\frac{\lambda_0}{\mu_0}} \,, \tag{42}$$

Solutions

As a result, ψ diminishes with the passing of time, while the scale factor goes like

$$a \propto t^{\frac{\lambda_0}{\mu_0}} . \tag{43}$$

It is also interesting to study the two limits $\mu_0 \to 0$ and $\mu_0 \to \infty$.

- In the former case the spin concentration (ψ) becomes constant and subsequently we get de Sitter-like expansion for the scale factor $a \propto e^{H_0 t}$. Hence we see that a constant spin distribution produces an exponential expansion.
- In the latter case (i.e., $\mu_0 \to \infty$) ψ essentially vanishes, resulting also in $H = 0$ and yielding, therefore, a static Universe. These cover the two extreme cases and for the rest in between we have the nice power-law solutions we derived above.

Conclusions/Further Prospects

- We have constructed the Perfect Cosmological Hyperfluid
- It can be further generalized by dropping the homogeneity assumption (Perfect Hyperfluid=Generalization of Perfect Fluid by taking into account the microstructure)
- The results apply also to Teleparallel Gravity (apart from MAG)
- We have derived Cosmological Solutions for Quadratic MAG
- What happens if we include quadratic Parity Odd Invariants as well?
- Connection to observations and bounds on hypermomentum variables?

...Thank you!!!