Lévy-Leblond Equation and Eisenhart-Duval lift in Koopman-von Neumann Mechanics

Zurab K. Silagadze

Outline of the talk

The talk is based on the paper Bikram Keshari Parida, Abhijit Sen, Shailesh Dhasmana, Zurab K. Silagadze, Lévy-Leblond Equation and Eisenhart-Duval lift in Koopman-von Neumann Mechanics, https://arxiv.org/abs/2308.16201
Published in Mod. Phys. Lett. A 38 (2023) 32n33, 2350149.

- Lévy-Leblond equation
- Koopmann-von Neumann mechanics
- Eisenhart-Duval lift
- A little about our work
- Concluding remarks

Foreword

“To the few who love me and whom I love – to those who feel rather than to those who think – to the dreamers and those who put faith in dreams as in the only realities – I offer this Book of Truths, not in its character of Truth-Teller, but for the Beauty that abounds in its Truth; constituting it true. To these I present the composition as an Art-Product alone: – let us say as a Romance; or, if I be not urging too lofty a claim, as a Poem.

What I here propound is true: – therefore it cannot die: – or if by any means it be now trodden down so that it die, it will «rise again to the Life Everlasting.»

Nevertheless it is as a Poem only that I wish this work to be judged after I am dead."

Edgar Allan Poe, Eureka. Poem in Prose (An Essay on the Material and Spiritual Universe) (1848).

Our quest for the secret of beauty

Reality is not always as harmonious as one might expect from our theories. But we continue our quest for the secret of beauty.

“Quest for the Secret of Beauty"
by Lado Gudiashvili (Georgian artist), 1942.

Did Friedman read Eureka?

Despite its rather naive and metaphysical premises, Eureka, this bizarre mixture of metaphysics, philosophy, poetry and science contains several brilliant ideas central to modern cosmology:

- version of the Big Bang, and the Universe evolving in time with the inflation of the primary atom at the beginning.
- resolution of Olbers' paradox (why the sky is dark at night).
- the application of the anthropic cosmological principle to explain why the Universe is so immensely large.
- the proposition of a multiverse with many causally unrelated universes, each with its own set of physical laws.

It is curious that, as his biographers testify, Poe was Friedman's favorite writer. "Did Friedmann read *Eureka*? It would be not serious to push this game too far."

A. Cappi, Edgar Allan Poe's Physical Cosmology, Q. J. Roy. Astron. Soc. 35 (1994), 177-192.

`https://adsabs.harvard.edu/full/1994QJRAS..35..177C`

Lévy-Leblond equation

- Around 1920, the riddle of anomalous Zeeman effect.
- Sommerfeld concluded that an additional quantum number J must exist corresponding perhaps to a hidden rotation.
- In 1921, Landé identified J as a total momentum and allowed half-integer M as a projection quantum number.
- In 1925, Uhlenbeck and Goudsmit first hypothesized intrinsic spin.
- Exclusion principle (1925) and Pauli equation (1927).

S. I. Tomonaga, The story of spin (University of Chicago Press, Chicago, 1998).

R. Milner, A Short History of Spin, `https://doi.org/10.22323/1.182.0003`

H. Schmidt-Böcking, G. Gruber, B. Friedrich, One hundred years ago Alfred Landé unriddled the Anomalous Zeeman Effect and presaged electron spin, `https://doi.org/10.1088/1402-4896/ac9c9b`

Spin - relativistic property?

- In 1928 Dirac proposed the equation that naturally incorporates spin.
- The Lie algebra of the Poincare group: six $\hat{J}_{\mu\nu} = -\hat{J}_{\nu\mu}$ generators of the Lorentz transformations, and four generators $\hat{P}_\mu$ of the translations.
- P_μ can be simultaneously diagonalized: $\hat{P}_\mu|p\rangle = p_\mu|p\rangle$.
- Little group is the subgroup of the Poincare group that leaves p_μ invariant.
- Spin and helicity arise as quantum numbers characterizing irreducible unitary representations of little groups: $SO(3)$ if $p_\mu = (m, 0, 0, 0)$, and $SE(2)$ if $p_\mu = (p, 0, 0, p)$.
- Spin is widely regarded as a relativistic property.

“The trouble with most folks isn’t so much their ignorance. It’s knowin’ so many things that ain’t so." Josh Billings (1818-1885).

J.-M. Lévy-Leblond, non-relativistic Particles and Wave Equations, Commun. Math. Phys. 6 (1967), 286. https://link.springer.com/article/10.1007/BF01646020

J.-M. Lévy-Leblond, Galilei Group and Nonrelativistic Quantum Mechanics, J. Math. Phys. 4 (1963), 776. https://doi.org/10.1063/1.1724319

“The main thing we want to emphasize is the naturalness of the concept of spin. Here this manifests itself in much the same way as in relativistic quantum mechanics. Accordingly, spin does not arise due to “relativistic effects” and is not a “consequence of the Dirac equation.” Even in non-relativistic quantum mechanics, spin should not be considered as an “extraneous hypothesis”, an “independent addition”, but, on the contrary, it immediately follows from first principles."

$$\left(\hat{p}_i\hat{p}_j\delta_{ij} - 2m\hat{E}\right)\psi(\vec{r},t) = 0.$$

$$\hat{\theta} = \frac{A}{c}\hat{E} + B_i\hat{p}_i + mcC$$

$\hat{\theta}^2 = \hat{p}_i\hat{p}_j\delta_{ij} - 2m\hat{E}$ (J.M. Wilkes, The Pauli and Lévy- Leblond equations, and the spin current density Eur. J. Phys. 41 (2020) 035402 https://arxiv.org/abs/1908.03276) or $\hat{\theta}\hat{\theta}' = \hat{p}_i\hat{p}_j\delta_{ij} - 2m\hat{E}$ (Lévy-Leblond).

$$A^2 = 0, \quad C^2 = 0, \quad AC + CA = -2,$$
$$AB_i + B_iA = 0, \quad CB_i + B_iC = 0,$$
$$B_iB_j + B_jB_i = 2\delta_{ij}.$$

- Lévy-Leblond Equation is obtained by linearizing the Schrödinger equation:

$$\vec{\sigma}\cdot(\hat{\vec{p}}-q\vec{A})\psi+2mc\chi=0$$
$$c\vec{\sigma}\cdot(\hat{\vec{p}}-q\vec{A})\chi+(\hat{E}-q\phi)\psi=0$$

- By eliminating χ, one can get the Pauli equation.
- Predicts the correct value for the intrinsic magnetic moment of a spin-1/2 particle ($g=2$).
- Little groups of the Galilei group also are characterized by s.
- Spin is not an intrinsically relativistic phenomena.

W.F. Eberlein, The Spin Model of Euclidean 3-Space, Am. Math. Monthly, 69 (1962), 587-598.
https://doi.org/10.1080/00029890.1962.11989933.

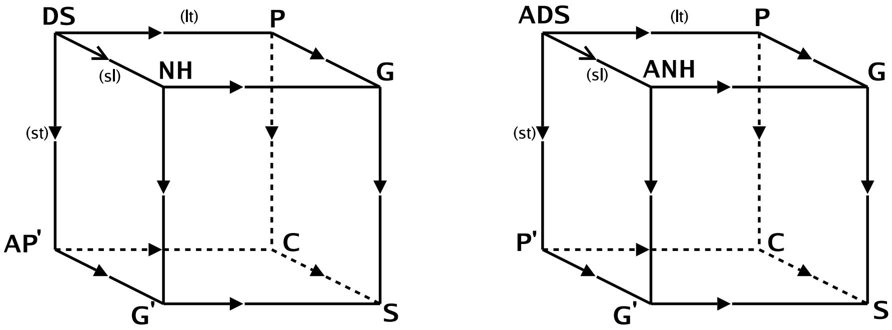
"It turns out that, contrary to the usual statement in the physical literature, electron spin then appears as a nonrelativistic effect."

$$c\vec{\sigma}\cdot(\hat{\vec{p}}-q\vec{A})\psi+(\hat{\mathcal{E}}+mc^2-q\phi)\chi=0$$
$$c\vec{\sigma}\cdot(\hat{\vec{p}}-q\vec{A})\chi+(\hat{\mathcal{E}}-mc^2-q\ phi)\psi=0$$

$$\mathcal{E}=mc^2+E,\ \ E,q\phi\ll mc^2.$$

"... thus recovering in almost a trivial way our wave equation, which appears very simply as the nonrelativistic limit of the Dirac equation. As a matter of fact, some authors have written down this equation when studying the Dirac equation and its nonrelativistic limit. However, they usually jump over to the Schrödinger equation which results from this, losing in the process many interesting features. On the contrary, it is most rewarding to pause a while at this stage, in order to exhibit the Galilean invariance of this nonrelativistic equation." (Lévy-Leblond).

Eleven kinematic groups



H. Bacry and J.-M. Lévy-Leblond, Possible kinematics, J. Math. Phys. 9 (1968), 1605-1614. https://doi.org/10.1063/1.1664490

Z.K. Silagadze, Relativity without tears, Acta Phys. Polon. B39 (2008), 811-885. https://arxiv.org/abs/0708.0929

De Broglie's relation between momentum and wavelength $p = \frac{h}{\lambda}$ contradicts the principle of relativity, since momentum depends on the choice of reference frame, but wavelength does not.

A. Landé, Quantum Fact and Fiction IV, Am. J. Phys. 43 (1975), 701–704. https://doi.org/10.1119/1.9717

J.-M. Lévy-Leblond, Quantum fact and classical fiction: Clarifying Landé's pseudo-paradox. https://doi.org/10.1119/1.10206

Wave function of a free nonrelativistic particle $\Psi(q) = e^{i(Et-\vec{p}\cdot\vec{r})}$ is not invariant under Galilean boosts $g: t' = t,\ \vec{r}' = \vec{r} - \vec{V}t$, $\vec{p}' = \vec{p} - m\vec{V},\ E' = E - \vec{p}\cdot\vec{V} + \frac{1}{2}mV^2$:

$$\Psi'(q) = e^{i\alpha(g;q)}\Psi(g^{-1}q).$$

$g_1(g_2q) = (g_1g_2)q$ requires

$$\alpha(g2; g1^{-1}q) - \alpha(g_1g_2; q) + \alpha(g_1; q) = \xi(g1, g2).$$

- Bargmann cocycle and the law of superselection by mass.
- Central extensions of symmetry groups.

Z.K. Silagadze, Relativity without tears, Acta Phys. Polon. B39 (2008), 811-885. https://arxiv.org/abs/0708.0929

Z.K. Silagadze, Relativistic mass and modern physics, Can. J. Phys. 92 (2014), 1643-1651. https://arxiv.org/abs/1103.6281

J. A. de Azcárraga, J. M. Izquierdo, Lie groups, Lie algebras, cohomology, and some applications in physics, Cambridge University Press, 1998.

Koopman-von Neumann mechanics

$$\frac{\partial\rho(q,p,t)}{\partial t}=\frac{\partial H_{cl}}{\partial q}\frac{\partial\rho}{\partial p}-\frac{\partial H_{cl}}{\partial p}\frac{\partial\rho}{\partial q}.$$

The classical wave function $\psi(q,p,t)=\sqrt{\rho(q,p,t)}$ obeys the same Liouville equation, which can be rewritten in Schrödinger-type form

$$i\frac{\partial\psi(q,p,t)}{\partial t}=\hat{L}\psi,\quad \hat{L}=i\left(\frac{\partial H_{cl}}{\partial q}\frac{\partial}{\partial p}-\frac{\partial H_{cl}}{\partial p}\frac{\partial}{\partial q}\right).$$

- It is possible to develop a formulation of classical mechanics in Hilbert space that completely resembles the quantum formalism, except that, of course, all interference effects are absent. Koopman 1931, von Neumann 1932.

D. Mauro, Topics in Koopman-von Neumann Theory, https://doi.org/10.48550/arXiv.quant-ph/0301172

- Ordinary axioms of quantum mechanics.
- $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$: unitary representation of a group of time shifts. According to Stone's theorem, there must exist a Hermitian generating operator with $i\frac{d|\Psi\rangle}{dt} = \hat{L}|\Psi\rangle$.
- Ehrenfest's theorem $\frac{d}{dt}\langle\hat{q}\rangle = \langle\frac{\hat{p}}{m}\rangle$, $\frac{d}{dt}\langle\hat{p}\rangle = -\langle\frac{d}{dq}U(\hat{q})\rangle$ requires

$$i[\hat{L},\hat{q}] = \frac{\hat{p}}{m}, \quad i[\hat{L},\hat{p}] = -\frac{d}{dq}U(\hat{q}).$$

- $[\hat{q},\hat{p}] = i\hbar$ -> quantum mechanics: $\hbar\hat{L} = \hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{q})$.
- $[\hat{q},\hat{p}] = 0$ -> we cannot construct $\hat{L}$ from only dynamic variables $\hat{q},\hat{p}$. To correct the situation, we introduce two additional Hermitian operators $\hat{\lambda}_q, \hat{\lambda}_p$, satisfying the conditions $[\hat{q},\hat{\lambda}_q] = i$, $[\hat{p},\hat{\lambda}_p] = i$. Then $\hat{L} = \frac{\hat{p}}{m}\hat{\lambda}_x - \frac{dU(\hat{q})}{dq}\hat{\lambda}_p$.

F. Wilczek, Notes on Koopman von Neumann Mechanics, and a Step Beyond. https://frankwilczek.com/2015/koopmanVonNeumann02.pdf

$$W(q,p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{i}{\hbar}py} \Psi^*(q+y/2,t)\Psi(q-y/2,t)dy.$$

$\hbar \to k\hbar,\ y = k\hbar\lambda_p,\ u = q - \frac{k\hbar\lambda_p}{2},\ v = q + \frac{k\hbar\lambda_p}{2}$:

$$W(q,p) = \sqrt{\frac{k\hbar}{2\pi}} \int e^{ip\lambda_p} \rho(u,v,t) d\lambda_p, \quad ; \rho(u,v,t) = \Psi^*(v)\Psi(u).$$

$$ik\hbar \frac{\partial \rho}{\partial t} = \left[\hat{H}_u - \hat{H}_v\right] \rho, \quad \hat{H}_u = \frac{(k\hbar)^2}{2m} \frac{\partial^2}{\partial u^2} + U(u).$$

This is reminiscent of the chiral decomposition method.
Generalized pseudo-differential Bopp operators:

$$\hat{u} = \hat{q} - \frac{k\hbar\hat{\lambda}_p}{2},\ \hat{v} = \hat{q} + \frac{k\hbar\hat{\lambda}_p}{2},\ \hat{p}_u = \hat{p} + \frac{k\hbar\hat{\lambda}_q}{2},\ \hat{p}_v = \hat{p} - \frac{k\hbar\hat{\lambda}_q}{2}.$$

$$[\hat{u}, \hat{p}_u] = ik\hbar, \quad [\hat{v}, \hat{p}_v] = -ik\hbar \quad k \to 0 \text{ means } [\hat{q}, \hat{p}] = 0.$$

Chiral decomposition method

The difference of Hamiltonians of two uncoupled one-dimensinal oscillators yield an interesting non-commutative system in the plane:

P. D. Alvarez, J. Gomis, K. Kamimura, M. S. Plyushchay, Anisotropic harmonic oscillator, non-commutative Landau problem and exotic Newton-Hooke symmetry, Phys. Lett. **B 659**, 906-912 (2008).
https://arxiv.org/abs/0711.2644

P. D. Alvarez, J. Gomis, K. Kamimura, and M. S. Plyushchay, (2+1)D Exotic Newton-Hooke Symmetry, Duality and Projective Phase, Annals Phys. **322** (2007) 1556-1586.
https://arxiv.org/abs/hep-th/0702014

P.-M. Zhang, P. A. Horvathy, Chiral Decomposition in the Non-Commutative Landau Problem, Annals Phys. 327 (2012) 1730–1743.
https://arxiv.org/abs/1112.0409.

$$\hat{H}_u - \hat{H}_v = \frac{k\hat{p}\hat{P}}{m} + U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - U\left(\hat{q} - \frac{k\hat{Q}}{2}\right), \quad \hat{\lambda}_q = \frac{\hat{P}}{\hbar}, \ \hat{\lambda}_p = -\frac{\hat{Q}}{\hbar}.$$

$$i\hbar\frac{\partial\Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{1}{k}U\left(\hat{q} + \frac{k\hat{Q}}{2}\right) - \frac{1}{k}U\left(\hat{q} - \frac{k\hat{Q}}{2}\right)\right]\Psi_{KvN},$$

where

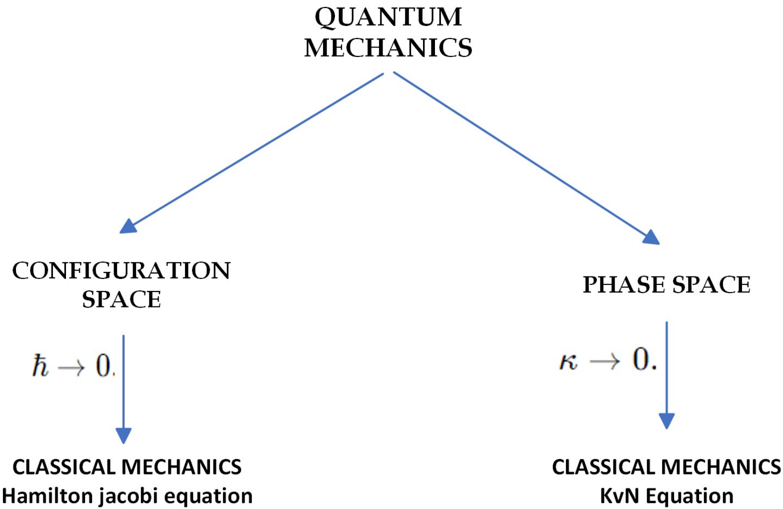
$$\Psi_{KvN}(q, Q, t) \sim \rho(u, v, t).$$

We have a well defined $k \to 0$ limit:

$$i\hbar\frac{\partial\Psi_{KvN}}{\partial t} = \left[\frac{\hat{p}\hat{P}}{m} + \frac{\partial U(q)}{\partial q}Q\right]\Psi_{KvN} = \hat{H}_{KvN}\Psi_{KvN}.$$

D.I. Bondar *et al.*, Operational dynamic modeling transcending quantum and classical mechanics, Phys. Rev. Lett. 109 (2012) 190403.
https://arxiv.org/abs/1105.4014

Comparison of $\hbar \to 0$ and $k \to 0$ limits



Sudarshan's interpretation

If we introduce $\hat{Q}$ and $\hat{P}$ operators as follows

$$\hat{Q} = i\hbar\frac{\partial}{\partial p}, \quad \hat{P} = -i\hbar\frac{\partial}{\partial q},$$

then the Liouville-Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi(q,p,t)}{\partial t} = \hat{H}\psi, \quad \hat{H} = \frac{\partial H_{cl}}{\partial q}\hat{Q} + \frac{\partial H_{cl}}{\partial p}\hat{P},$$

and it can be interpreted as the Schrödinger equation in the (q,p)-representation (with diagonal operators q and p) of a genuine quantum system with two pairs of canonical variables (q,P) and (Q,p).

E. C. G. Sudarshan, Interaction between classical and quantum systems and the measurement of quantum observables, Pramana 6(3) (1976), 117. https://link.springer.com/article/10.1007/BF02847120

Quantum Mechanics Free Subsystems (QMFS)

Let us assume that the Hamiltonian of the quantum system is equal to

$$\hat{H} = f(q,p,t)\hat{P} + g(q,p,t)\hat{Q} + h(q,p,t),$$

where $f(q,p,t)$, $g(q,p,t)$, $h(q,p,t)$ are arbitrary functions, and q, P and Q, p represent are two pairs of quantum mechanical conjugate variables that obey canonical commutation relations. Then the Heisenberg equations of motion for the commuting variables q, p

$$\frac{dq}{dt} = \frac{\partial H}{\partial P} = f(q,p,t), \quad \frac{dp}{dt} = -\frac{\partial H}{\partial Q} = -g(q,p,t),$$

do not contain "hidden" variables $\hat{Q}, \hat{P}$ and will correspond to classical Hamiltonian dynamics if there exists a classical Hamiltonian function $H_{cl}(q,p,t)$ such that

$$f(q,p,t) = \frac{\partial H_{cl}}{\partial p}, \quad g(q,p,t) = \frac{\partial H_{cl}}{\partial q}.$$

Experimental implementation of QMFS

M. Tsang, C. M. Caves, Evading quantum mechanics: Engineering a classical subsystem within a quantum environment, Phys. Rev. X 2 (2012), 031016. https://arxiv.org/abs/1203.2317 A pair of positive and negative mass oscillators can be used for this purpose. The quantum Hamiltonian in this case has the form

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2 - \frac{p_2^2}{2m} - \frac{1}{2}m\ omega^2 q_2^2.$$

In terms of new canonical variables

$$q = q_1 + q_2, \;\; Q = \frac{1}{2}(q_1 - q_2), \;\; p = p_1 - p_2, \;\; P = \frac{1}{2}(p_1 + p_2),$$

The Hamiltonian takes the form $H = \frac{pP}{m} + m\omega^2 qQ$, and is a KvN-type Hamiltonian.
Sidney Coleman: "The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction."

KvN mechanics is realized in QMFS

- Similarity of the Sudarshan interpretation of the KvN mechanics with the idea of QMFS is obvious.
- (q, p) subsystem of KvN mechanics is nothing more than QMFS.
- Resumption of interest in KvN mechanics was caused by the need to create suitable formalism for hybrid classical-quantum systems.
- The identification of quantum-mechanics-free subsystems with Sudarshan's interpretation of KvN mechanics, combined with the fact that such systems were actually implemented experimentally, makes the KvN mechanics, in a sense, engineering science.

Z.K. Silagadze, Evading Quantum Mechanics à la Sudarshan: quantum-mechanics-free subsystem as a realization of Koopman-von Neumann mechanics, https://arxiv.org/abs/2308.08919. Published in Foundations of Physics 53 (2023), 92.

Quantum gravity destroys classicality?

- Modification of quantum mechanics, expected from quantum gravity, can lead to deformation of classical mechanics (O.I Chashchina, A. Sen, Z.K. Silagadze, On deformations of classical mechanics due to Planck-scale physics, Int. J. Mod. Phys. D29 (2020), 2050070 https://arxiv.org/abs/1902.09728).
- This deformation actually destroys the classicality if Sudarshan's views on KvN mechanics are taken seriously.
- You are not required to accept the Sudarshan interpretation in order to develop the KvN mechanics.
- However, we now see that the existence of quantum-mechanics-free subsystems indicates that we should take Sudarshan's interpretation of KvN mechanics seriously.
- Therefore, we expect that, due to the universal nature of gravity, if the effects of quantum gravity do modify quantum mechanics, these effects will destroy the classical dynamics in QMFS.

- Path integral formulation of KvN mechanics introduces two Grassmannian partners of the time.
- Standard quantization rules become equivalent to freezing to zero of these Grassmannian partners of time.
- The formulation of KvN mechanics via path integrals has many features in common with Witten's topological field theories.

A.A. Abrikosov (Jr.), E. Gozzi, Quantization and Time, Nucl. Phys. B Proc. Suppl. 88 (2000), 369.
https://arxiv.org/aabs/quant-ph/9912050

E. Gozzi, M. Reuter, Classical Mechanics as a Topological Field Theory, Phys. Lett. B 240 (1990), 137.
https://doi.org/10.1016/0370-2693(90)90422-3

Eisenhart-Duval lift

Embedding nonrelativistic physics inside a gravitational wave

Eisenhart theorem (1929):
Dynamical trajectories of non-relativistic (NR) mechanics can always be lifted to geodesics of a specific relativistic spacetime with one dimension more. Conversely, to any geodesic of this specific class of spacetimes corresponds a solution of a NR dynamical system.

- The metric is uniquely determined by the form of the NR Lagrangian.
- Relativistic spacetime has a metric with Lorentz signature and carries a covariantly constant null vector (Bargmann structure, Duval, 1985).
- Bargmann space is in fact the space-time of a plane gravitational wave in 5-dimensions.

Plato cave allegory (Minguzzi, 2006)

X. Bekaert, K. Morand, Embedding nonrelativistic physics inside a gravitational wave, Phys. Rev. D 88 (2013), 063008.
https://arxiv.org/abs/1307.6263

The simplest way to explain Eisenhart-Duval lift is to use Hamiltonian approach. The first step is to promote time t to a dynamical variable:

$$H=\frac{1}{2m}\sum_{i,j=1}^{n}h^{ij}(q)p_ip_j+V(q,t)\to\tilde{H}=p_t+H(q,p,t)=0.$$

The main idea behind the Eisenhart lift is to introduce a new momentum p_s conjugate to a dummy configuration space variable s to make the Hamiltonian homogeneous in canonical momenta and turn it into a geodesic Hamiltonian (homogeneous quadratic function of momenta):

$$\mathcal{H}=\frac{1}{2m}\sum_{i,j=1}^{n}h^{ij}(q)p_ip_j+\frac{1}{m^2}\,p_s^2\,V(q,t)+\frac{1}{m}\,p_sp_t=\frac{1}{2m}\sum_{A,B=1}^{n+2}g^{AB}p_Ap_B.$$

The constraint $\mathcal{H}=0$ can be interpreted as a mass-shell condition for a massless particle in space-time with a Brinkmann-type metric

$$dS^2=\sum_{i,j=1}^{n}h_{ij}\,dq^i\,dq^j+2ds\,dt-2\frac{V(q,t)}{m}\,dt^2.$$

The massless Klein-Gordon equation in general metric is given by

$$\Box\phi = \frac{1}{\sqrt{-g}}\,\partial_A\left(\sqrt{-g}\,g^{AB}\partial_B\phi\right) = 0,$$

which, after the field transformation (null-reduction)

$$\phi(q,t,s) = e^{is}\varphi\,(q,t)\,,$$

is reduced to the Schrödinger equation

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi + V\varphi.$$

The Schrödinger equation can be considered as a null-reduction (reduction in the s-direction) of the Klein-Gordon equation in the Eisenhart-Duval metric background.

(2+1)D Lévy-Leblond equation can be derived from the massless (3+1)D Dirac equation by lightlike reduction:

C. Duval, P. A. Horváthy, L. Palla, Spinors in non-relativistic Chern-Simons electrodynamics. Annals Phys. 249 (1996) 265-297.
https://arxiv.org/abs/hep-th/951011

Eisenhart-Duval lift in KvN mechanics

The simplest way to geometrize the KvN mechanics is to begin from the KvN Hamiltonian and consider it as describing classical (not KvN) system:

$$H = \frac{pP}{m} + \frac{\partial V}{\partial q} Q.$$

Homogenizing this Hamiltonian, we get

$$\mathcal{H} = \frac{pP}{m} + \frac{\partial V}{\partial q} Q \frac{p_s^2}{m^2} + \frac{1}{m} p_s p_t,$$

which corresponds to the inverse metric

$$g^{qQ} = g^{Qq} = 1, \;\; g^{st} = g^{ts} = 1, \;\; g^{ss} = \frac{2}{m} \frac{\partial V}{\partial q} Q,$$

all other components being zero. Inverting g^{AB} to calculate the metric tensor g_{AB}, we get the corresponding Eisenhart metric

$$dS^2 = 2dq\, dQ + 2dt\, ds - \frac{2Q}{m} \frac{\partial V(q)}{\partial q} dt^2.$$

Curved space KG for the massless scalar field $\chi(t,s,q,Q)$ for KvN Eisenhart-Duval metric is

$$\frac{\partial^2\chi}{\partial q\partial Q}+\frac{Q}{m}\frac{\partial V}{\partial q}\frac{\partial^2\chi}{\partial s^2}+\frac{\partial^2\chi}{\partial t\partial s}=0,$$

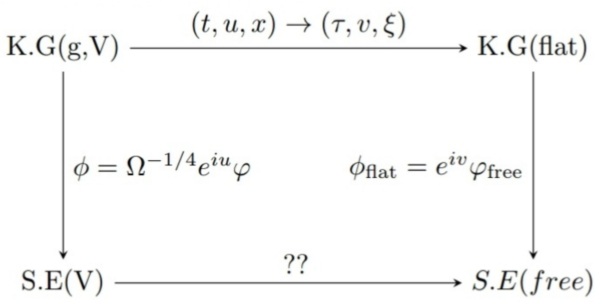
which after the field redefinition

$$\chi(t,s,q,Q)=e^{ims}\psi_{KvN}(t,q,Q),$$

reduces to the equation of the form

$$i\frac{\partial\psi_{KvN}}{\partial t}=\left(Q\frac{\partial V}{\partial q}-\frac{1}{m}\frac{\partial^2}{\partial q\partial Q}\right)\psi_{KvN},$$

which is the KvN equation in the (q,Q)-representation for the classical Hamiltonian $H=\frac{p^2}{2m}+V(q)$.



Conditions for flatness: the Cotton tensor ($d = 3$) or the Weyl tensor ($d > 3$) vanishes.

$$C_{\mu\nu\lambda} = \nabla_\lambda R_{\mu\nu} - \nabla_\nu R_{\mu\lambda} + \frac{1}{4}\left(g_{\mu\lambda}\nabla_\nu R - g_{\mu\nu}\nabla_\lambda R\right).$$

C. Duval, P.A. Horváthy, L. Palla, Conformal Properties of Chern-Simons Vortices in External Fields. Phys. Rev. D50 (1994), 6658-6661.
https://arxiv.org/abs/hep-th/9404047

$$V = \frac{1}{2}A(t)\vec{x}^2 + B\vec{(t)} \cdot \vec{x} + D(t).$$

Linear potential and free particle

Uniform gravitational field:

$$t=\tau, \quad x=\xi-\frac{g\tau^2}{2}, \;\; u=v+g\xi\tau-\frac{g^2\tau^3}{3},$$

$$\varphi_{grv}(t,x)=e^{-i\left[g\xi\tau-\frac{g^2\tau^3}{3}\right]}\varphi_{free}(\tau,\xi).$$

S. Dhasmana, A. Sen, Z.K. Silagadze, Equivalence of a harmonic oscillator to a free particle and Eisenhart lift, Annals Phys. 434 (2021) 168623. https://arxiv.org/abs/2106.09523

Phenomena of neutron interference in the presence of a weak gravitational potential: R. Colella, A.W. Overhauser, S.A. Werner, Observation of gravitationally induced quantum interference, Phys. Rev. Lett. 34 (1975), 1472-1474. https://doi.org/10.1103/PhysRevLett.34.1472

Harmonic oscillator and free particle

Transformation between harmonic oscillator and free particle:

$$x=\frac{\xi+c_2\,\omega_0\tau+c_1}{\sqrt{1+\omega_0^2\tau^2}},\quad t=\frac{1}{\omega_0}\tan^{-1}\left(\omega_0\tau\right),\quad u=$$

$$v+\frac{\omega_0^2\xi^2\tau}{2(1+\omega_0^2\tau^2)}-\frac{(c_2-c_1\,\omega_0\tau)\,\omega_0}{1+\omega_0^2\tau^2}\,\xi+\frac{(c_1^2-c_2^2)\omega_0^2\tau-2c_1c_2\,\omega_0}{2(1+\omega_0^2\tau^2)}+c_3,$$

whereas the wave function transforms as follows:

$$\varphi_{H.O}(t,x)=(1+\omega_0^2\tau^2)^{1/4}\times$$

$$e^{-i\left(\frac{\omega_0^2\xi^2\tau}{2(1+\omega_0^2\tau^2)}-\frac{(c_2-c_1\,\omega_0\tau)\omega_0}{(1+\omega_0^2\tau^2)}\xi+\frac{(c_1^2-c_2^2)\omega_0^2\tau-2c_1c_2\omega_0}{2(1+\omega_0^2\tau^2)}+c_3\right)}\varphi_{free}(\tau,\xi)$$

S. Dhasmana, A. Sen, Z.K. Silagadze, Equivalence of a harmonic oscillator to a free particle and Eisenhart lift, Annals Phys. 434 (2021) 168623.
https://arxiv.org/abs/2106.09523

The case of Koopmann von Neumann mechanics

For a harmonic oscillator, the coordinate transformation has the form

$$t = \tan^{-1}(\tau), \quad u = \frac{-\eta}{\sqrt{\tau^2+1}}, \quad v = \frac{-\xi}{\sqrt{\tau^2+1}},$$
$$s = \zeta + \frac{1}{2(\tau^2+1)}\left[\left(\eta^2-\xi^2\right)\tau\right]. \quad q = \frac{v+u}{2}, \quad Q = u - v.$$

Corresponding transformation of the KvN wave function

$$\psi_{HO} = \sqrt{1+\tau^2}\ \exp\left[\frac{im\left[\left(\xi^2-\eta^2\right)\tau\right]}{2(\tau^2+1)}\right]\psi_{free}.$$

For a linear potential, the coordinate transformation has the form

$$t = \tau,\ u = \eta - \frac{1}{2}g\tau^2,\ v = \xi - \frac{1}{2}g\tau^2,\ s = \zeta + (\eta-\xi)g\tau.$$

Corresponding transformation of the KvN wave function

$$\psi_{Linear}(q,Q,t) = e^{-iQg\tau}\psi_{free}\left(q + \frac{1}{2}gt^2, Q, t\right),$$

is unitary and represents Einstein's equivalence principle in KvN mechanics.

A. Sen, B.K. Parida, S. Dhasmana, Z.K. Silagadze, Eisenhart lift of Koopman-von Neumann mechanics, J. Geom. Phys. 185 (2023), 104732 https://arxiv.org/abs/2207.05073

Latin indices indicate the tensor components in the local orthonormal frame (locally inertial frame). Curved space massless Dirac equation is

$$i\, e_a^{\ \mu}\, \gamma^a\, \nabla_\mu \Psi = 0,$$

where the co-variant derivatives of the Dirac spinor field $\Psi(Q, q, s, t)$ is

$$\nabla_\mu \Psi = \partial_\mu \Psi + i\, \Omega_{\mu ab}\, \Sigma^{ab} \Psi,$$

$$\Sigma^{ab} = -\frac{i}{8}\, \left[\gamma^a, \gamma^b\right], \tag{1}$$

and the Christoffel symbols and spin-connection coefficients are given by the usual expressions

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right), \quad \Omega_{\mu\ \ b}^{\ \ a} = e_b^{\rho}\, e_\nu^{a}\, \Gamma^{\nu}_{\ \mu\rho} - e_b^{\nu}\, \partial_\mu e_\nu^{a}.$$

Vierbein fields

The vierbein $e_a^{\ \mu}$ satisfies $e_a^{\ \mu} g_{\mu\nu} e_b^{\ \nu} = \eta_{ab}$ and is defined up to a $SO(2,2)$ transformation. Can be chosen as

$$e_a^{\ \mu} = \begin{pmatrix} 0 & 0 & \frac{1}{2} + \frac{Q}{m}\frac{\partial V(q,t)}{\partial q} & 1 \\ -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{Q}{m}\frac{\partial V(q,t)}{\partial q} & -1 \\ 1 & -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2Q}{m}\frac{\partial V(q,t)}{\partial q} \end{pmatrix}$$

Constant gamma matrices

Constant gamma matrices in the tetrad frame satisfying the Clifford algebra

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2\,\eta^{ab}$$

can be given as

$$\gamma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\qquad \gamma^2=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$

$$\gamma^3=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix},\qquad \gamma^4=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Christoffel symbols and spin-connection coefficients

$$\Gamma^q_{\ tt} = -\Gamma^s_{\ Qt} = -\Gamma^s_{\ tQ} = \frac{1}{m}\frac{\partial V}{\partial q},$$

$$\Gamma^Q_{\ tt} = -\Gamma^s_{\ qt} = -\Gamma^s_{\ tq} = \frac{Q}{m}\frac{\partial^2 V}{\partial q^2}, \quad \Gamma^s_{\ tt} = -\frac{Q}{m}\frac{\partial^2 V}{\partial q \partial t},$$

$$\Omega_{t12} = -\Omega_{t21} = \Omega_{t23} = -\Omega_{t32} = \frac{1}{m}\frac{\partial V}{\partial q} + \frac{Q}{2m}\frac{\partial^2 V}{\partial q^2},$$

$$\Omega_{t34} = -\Omega_{t43} = \Omega_{t41} = -\Omega_{t14} = \frac{1}{m}\frac{\partial V}{\partial q} - \frac{Q}{2m}\frac{\partial^2 V}{\partial q^2}.$$

Accordingly, we calculate

$$i\,\Omega_{tab}\Sigma^{ab} = \frac{1}{2}(\gamma^1 - \gamma^3)(\Omega_{t12}\gamma^2 - \Omega_{t34}\gamma^4).$$

On the other hand, $e_a^{\ t}\gamma^a = \gamma^1 - \gamma^3$, and since $(\gamma^1 - \gamma^3)^2 = 0$, we see that the spin connection does not contribute to the Dirac equation.

$$-2\frac{\partial\psi_3}{\partial Q}+2\left(\frac{\partial\psi_4}{\partial t}+\frac{Q}{m}\frac{\partial V(q,t)}{\partial q}\frac{\partial\psi_4}{\partial s}\right)=0,\quad \frac{\partial\psi_3}{\partial s}+\frac{\partial\psi_4}{\partial q}=0,$$
$$-\frac{\partial\psi_1}{\partial q}+2\left(\frac{\partial\psi_2}{\partial t}+\frac{Q}{m}\frac{\partial V(q,t)}{\partial q}\frac{\partial\psi_2}{\partial s}\right)=0,\quad \frac{\partial\psi_1}{\partial s}+2\frac{\partial\psi_2}{\partial Q}=0.$$

The null reduction of the Dirac equation is achieved by requiring

$$\Psi(Q,q,s,t)=\Phi(Q,q,t)e^{ims}=\begin{pmatrix}\phi_1(Q,q,t)\\ \phi_2(Q,q,t)\\ \phi_3(Q,q,t)\\ \phi_4(Q,q,t)\end{pmatrix}e^{ims}.$$

Then we obtain from

$$-\frac{\partial\phi_3}{\partial Q}+\frac{\partial\phi_4}{\partial t}+i\,Q\frac{\partial V(q,t)}{\partial q}\phi_4=0,\qquad \phi_3=\frac{i}{m}\frac{\partial\phi_4}{\partial q},$$
$$-\frac{\partial\phi_1}{\partial q}+2\left(\frac{\partial\phi_2}{\partial t}+i\,Q\frac{\partial V(q,t)}{\partial q}\phi_2\right)=0,\quad \phi_1=\frac{2i}{m}\frac{\partial\phi_2}{\partial Q}.$$

KvN Lévy-Leblond Equation

The Lévy-Leblond equation in the KvN case can be written as

$$-\left[\frac{\sigma_1}{2}\left(\hat{p}+\frac{\hat{P}}{2}\right)+i\,\frac{\sigma_2}{2}\left(\hat{p}-\frac{\hat{P}}{2}\right)\right]\chi+Q\frac{\partial V(q,t)}{\partial q}\xi=\hat{E}\xi,$$

$$\chi=-\frac{1}{m}\left[\sigma_1\left(\hat{p}+\frac{\hat{P}}{2}\right)+i\,\sigma_2\left(\hat{p}-\frac{\hat{P}}{2}\right)\right]\xi.$$

Another version of the KvN Lévy-Leblond equation:

$$-\left[\sigma_+\,\hat{p}+\sigma_-\,\frac{\hat{P}}{2}\right]\chi+Q\frac{\partial V(q,t)}{\partial q}\xi=\hat{E}\,\xi,$$

$$\chi=-\frac{2}{m}\left[\sigma_+\,\hat{p}+\sigma_-\,\frac{\hat{P}}{2}\right]\xi.$$

$\chi=\begin{pmatrix}\phi_1\\ \phi_3\end{pmatrix}$, $\xi=\begin{pmatrix}\phi_4\\ \phi_2\end{pmatrix}$, $\hat{p}=-i\dfrac{\partial}{\partial Q}$, $\hat{P}=-i\dfrac{\partial}{\partial q}$, $\hat{E}=i\dfrac{\partial}{\partial t}$,
$\sigma_+=\dfrac{\sigma_1+i\,\sigma_2}{2}$ and $\sigma_-=\dfrac{\sigma_1-i\,\sigma_2}{2}$.

Conclusions

- A general holonomic conservative system in classical dynamics with d degrees of freedom is geometrically described by the Eisenhart-Duval lift in terms of the geodesics of the Lorentzian metric in the $(d+2)$-dimensional space-time.
- For treating time dependent dynamical systems and their symmetries, this geometric perspective is particularly convenient.
- The same geometric perspective provided by the Eisenhart-Duval lift can also be used in quantum theory, since null reduction of the massless KG equation from Eisenhart-Duval space-time leads to the Schrödinger equation, and a similar null reduction of the massless Dirac equation gives the Lévy-Leblond equation.
- The Eisenhart-Duval toolkit can be applied to KvN mechanics as well, much like the quantum case. Namely, this geometrical view on the KvN mechanics can be extended to the case of a non-relativistic spin described by the Lévy-Leblond equation.