Ultracompact stars in the light of minimal geometric deformation

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MGTs have been the subject of great interest in cosmology and provide a convincing way for settling the issue of cosmic acceleration. The concept that gravity is not described precisely by GR but rather by some alternative theories has been viewed for several years.
There are various ways to modify GR incorporating "quadratic Lagrangian", consisting of second order curvature invariants such as R^2, $R_{\alpha\beta}R^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$.
Therefore, the general modification of GR action is of the form

$$\begin{aligned} \mathcal{I} &= \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}f(R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, ..) \\ &+ \int dx^4\sqrt{-g}\mathcal{L}_m(g_{\alpha\beta,}\Psi_m). \end{aligned} \quad (1)$$

Modified Gravitational Theories

Such theories involve the higher order derivatives and allow the dynamical equations to be higher than second order.

Modifications of gravity have received significant attention in order to explain the cosmic acceleration. In this respect, a particularly interesting modification is to replace the linear dependence of scalar curvature with the more generic function and resulting action is named as $f(R)$ gravity [Nojiri, S. and Odintsov, S.D.: Int. J. Geom. Methods Mod. Phys. **4**(2007)115.].

$f(\mathcal{T})$ gravity ($\mathcal{T}$ being the torsion), $f(R, T)$ gravity (T being the trace of $T_{\mu\nu}$), $f(R, T, Q)$ gravity ($Q = R_{\mu\nu}T^{\mu\nu}$), Scalar Tensor theories, Brans-Dicke theory, Gauss-Bonnet gravity etc.

Minimal and Non-minimal Coupling:

MGTs are constructed by incorporating the geometric part whereas matter contribution is considered as additional term in Lagrangian. Nevertheless one can put further modification by introducing direct coupling between matter and curvature components; such theory is named as non-minimally coupled gravity.

Such couplings were initially proposed in [Nojiri, S. and Odintsov, S.D.: Phys. Lett. B **599**(2004)137][Allemandi, G. et al.: Phys. Rev. D **72**(2005)063505].

Bertolami et al. [Phys. Rev. D **75**(2007)104016] put a new twist on $f(R)$ gravity by considering the Lagrangian as a function of scalar curvature and explicit coupling between scalar curvature term and matter Lagrangian density. The action of $f(R)$ gravity with a non-minimal gravitational coupling to matter is given by

$$\mathcal{S} = \int \sqrt{-g} dx^4 \{ \frac{1}{2\kappa^2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \}. \qquad (2)$$

The parameter λ characterizes the strength of non-minimal coupling of $f_2(R)$ with matter Lagrangian.

A more generalized form of $f(R)$ gravity is suggested in [Harko, T. and Lobo, F.S.N.: Eur. Phys. J. C **70**(2010)373], where the Lagrangian is an arbitrary function of Ricci sclar R and of matter Lagrangian $\mathcal{L}_m$ *i.e.*, $\mathcal{L} = f(R, \mathcal{L}_m)$.

$$\mathcal{S} = \frac{1}{2\kappa} \int \sqrt{-g} dx^4 f(R, \mathcal{L}_m) + S_{(m)}(g_{\mu\nu}, \psi_m),$$

In [Harko et al.: Phys. Rev. D **89**, 124036 (2014)] developed a modified $f(\mathcal{T})$ model, allowing a non-minimal coupling between the torsion scalar and the matter Lagrangian, whose action is given by

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \sqrt{-g}dx^4 e\{\mathcal{T} + f_1(\mathcal{T}) + [1+\alpha f_2(\mathcal{T})]\mathcal{L}_m\}, \quad (3)$$

where $f_i(\mathcal{T})(i=1,2)$ are arbitrary functions of torsion scalar and α is the coupling parameter.

Modified Gravitational Theories

In [Kofinas and Saridakis Phys. Rev. D **90**(2014)084044], Kofinas and Saridakis proposed a modified theory involving both torsion scalar T and the teleparallel equivalent of Gauss-Bonnet term T_G as basic ingredient, defined by the following action

$$S = \frac{1}{2\kappa^2} \int_M d^4 x e f(T, T_G) + S_m, \tag{4}$$

where $e = det(e^a_\mu) = \sqrt{|g|}$ and $\kappa^2 = 8\pi G$. In some certain limits of the function $f(T, T_G)$, other theories like GR, TEGR, Einstein-Gauss-Bonnet theory etc. can be discussed.

In 2011, Harko et al. [Phys. Rev. D **84**(2011)024020] presented a new modification of Einstein Lagrangian by introducing an arbitrary function of scalar curvature R and trace of the energy-momentum tensor T. The action of $f(R,T)$ theory of gravity is given by

$$\mathcal{S} = \frac{1}{2\kappa^2}\int \sqrt{-g}dx^4 f(R,T) + S_{(m)}(g_{\mu\nu},\psi_m), \quad (5)$$

- In general $f(R, T)$ gravity models, the matter energy momentum tensor is not covariantly conserved, it follows that motion of test particles is nongeodesic and an extra acceleration is always present due to the coupling between matter and geometry [Phys. Rev. D **84**(2011)024020]. This situation is similar to the case of $f(R, \mathcal{L}_m)$ gravity and $f(R)$ gravity with non-minimal coupling to matter.

- In this modified gravity cosmic acceleration may result not only due to geometrical contribution to the total cosmic energy density but it also depends on matter contents. This theory can be applied to explore several issues of current interest and may lead to some major differences.

Field equations of $f(R, T)$ gravity

$$R_{\mu\nu} f_R(R,T) - \frac{1}{2} g_{\mu\nu} f(R,T) + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f_R(R,T) = 8\pi G T_{\mu\nu} - f_T(R,T) T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}, \quad (6)$$

$\Box : \nabla_\mu \nabla^\mu$, ∇_μ: the covariant derivative associated with the Levi-Civita connection of the metric, f: arbitrary function of R and T; $f_R : \partial f/\partial R$, $f_T : \partial f/\partial T$, $\Theta_{\mu\nu} = \frac{g^{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$.

In $f(R, T)$ gravity, the divergence of the energy-momentum tensor is not covariantly conserved and is given by

$$\nabla^\alpha T_{\alpha\beta} = \frac{f_T}{\kappa^2 - f_T}\left[(T_{\alpha\beta} + \Theta_{\alpha\beta})\nabla^\alpha \ln f_T + \nabla^\alpha \Theta_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \nabla^\alpha T\right] \quad (7)$$

- Ovalle [Ovalle, J. (2017). Phys. Rev. D, **95**, 104019] proposed the idea of gravitational decoupling by means of Minimal Geometric Deformation (MGD)— highly versatile tool

- Seed matter content $T_{\alpha\beta}$ + New gravitational source

$$T_{\alpha\beta} \longrightarrow \tilde{T}^{(n)}_{\alpha\beta} = T_{\alpha\beta} + \tilde{\beta}^{(n)}\,\hat{T}^{(n)}_{\alpha\beta}. \quad (8)$$

Here, $\tilde{\beta}$ is a dimensionless coupling parameter.

- After combining two different sources, a system of field equations is formulated.

- One needs to introduce transformation onto the metric potentials* of $\mathcal{M}^-$

$$\chi(r) \mapsto \zeta(r) + \tilde{\beta}k(r), \quad e^{-\psi(r)} \mapsto \eta(r) + \tilde{\beta}h^*(r). \quad (9)$$

* **The inner geometry of the compact structure is defined by**

$$ds^2 = -e^{\chi(r)}dt^2 + e^{\psi(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

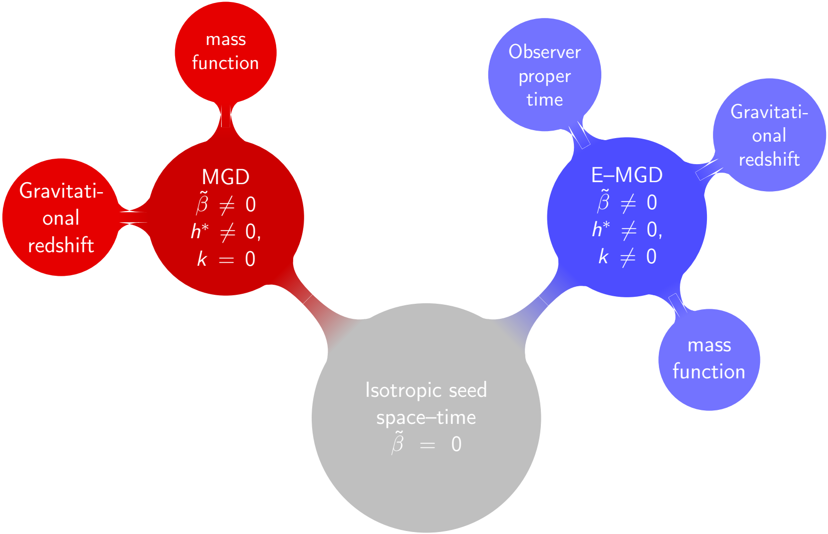


Figure: From to non–deformed space–time to a minimally deformed/completely deformed space–time by MGD/e–MGD, respectively

$f(R, T) + MGD$ formalism

The action for the $f(R, T)$ theory of gravity with L_φ as Lagrangian density for an additional source $\varphi_{\alpha\beta}$ assumes the form

$$I = \frac{1}{2\kappa}\int f(R, T)\sqrt{-g}d^4x + \int [L_m + \tilde{\beta}L_\varphi]\sqrt{-g}d^4x. \tag{10}$$

Varying the gravitational action with respect to $g_{\alpha\beta}$, yields

$$\begin{aligned} G_{\alpha\beta} &= \frac{1}{f_R}\Big[(8\pi + f_T)T_{\alpha\beta} + (\frac{f - Rf_R}{2})g_{\alpha\beta} + (\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box)f_R \\ &\quad + Pg_{\alpha\beta}f_T + 8\pi\tilde{\beta}\varphi_{\alpha\beta}\Big] = \tilde{T}_{\alpha\beta}. \end{aligned} \tag{11}$$

Substituting the expressions evaluating from usual definition of $T_{\alpha\beta}$ for perfect fluid into the field equations (11), one gets

$$\left[\frac{1}{r^2}+e^{-\psi}\left(\frac{\psi'}{r}-\frac{1}{r^2}\right)\right]f_R = \tilde{\rho}, \tag{12}$$

$$\left[e^{-\psi}\left(\frac{\chi'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}\right]f_R = \tilde{P}_r, \tag{13}$$

$$e^{-\psi}\left[\frac{\chi''}{2}-\frac{\psi'}{2r}+\frac{\chi'}{2r}-\frac{\psi'\chi'}{4}+\frac{\chi'^2}{4}\right]f_R = \tilde{P}_t, \tag{14}$$

where,

$$\tilde{\rho} = (8\pi+f_T)\rho-\frac{f-Rf_R}{2}+pf_T-e^{-\psi}\left[f_R''-f_R'\left(\frac{\psi'}{2}-\frac{2}{r}\right)\right]+8\pi\tilde{\beta}\varphi_0^0, \tag{15}$$

$$\tilde{P}_r = P+\frac{f-Rf_R}{2}+e^{-\psi}\left(\frac{\chi'}{2}+\frac{2}{r}\right)f_R'-8\pi\tilde{\beta}\varphi_1^1, \tag{16}$$

$$\tilde{P}_t = P+\frac{f-Rf_R}{2}-e^{-\psi}\left[\left(\frac{\psi'-\chi'}{2}-\frac{1}{r}\right)f_R'-f_R''\right]-8\pi\tilde{\beta}\varphi_2^2. \tag{17}$$

$f(R,T) = R + \lambda T, \quad \lambda$ is a dimensionless coupling constant

1. For the complex $f(R,T)$ functions, the new emerging sectors will be too complex to solve analytically.
2. The second issue arises when one deals with matching conditions across the boundary.

The pure matter sector given in (15)-(17) has the form

$$\bar{\rho} = \rho + \frac{\lambda}{16\pi}(3\rho - P) + \tilde{\beta}\varphi_0^0, \tag{18}$$

$$\bar{P}_r = P - \frac{\lambda}{16\pi}(\rho - 3P) - \tilde{\beta}\varphi_1^1, \tag{19}$$

$$\bar{P}_t = P - \frac{\lambda}{16\pi}(\rho - 3P) - \tilde{\beta}\varphi_2^2, \tag{20}$$

Mass Function

The **mass function** is also affected in this formulation and gravitational mass of stellar structure takes the form as

$$\tilde{m}(r) = 4\pi \int_0^r \rho r^2 dr + \frac{\lambda}{4} \int_0^r (3\rho - P) r^2 dr + +4\pi\tilde{\beta} \int_0^r \varphi_0^0(r) r^2 dr. \quad (21)$$

- This expression contains all the contributions coming from MGD, and $f(R, T)$ gravity.

  Here, it is important to mention that the above expression for the mass $\tilde{m}(r)$ is going beyond the pure GR scope. This scenario facilitates the construction of more compact objects, at least on theoretical grounds. Moreover, the case for the limits $\lambda = 0$ and $\tilde{\beta} = 0$, yields the usual mass function in the arena of GR.

- By setting $\tilde{\beta} = 0$, we obtain mass function for $f(R, T)$ gravity.

Splitting of $f(R,T)$ Field Equations Through MGD

Plugging the linear transformation in field equations (12)-(14) and comparing the expressions for $\tilde{\beta}=0$, one gets

$$8\pi\rho+\frac{\lambda}{2}(3\rho-P) = \frac{1}{r^2}-\frac{\eta}{r^2}-\frac{\eta'}{r}, \tag{22}$$

$$8\pi P+\frac{\lambda}{2}(3P-\rho) = \eta\left(\frac{\chi'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \tag{23}$$

$$8\pi P+\frac{\lambda}{2}(3P-\rho) = \eta\left(\frac{\chi''}{2}+\frac{(\chi')^2}{4}+\frac{\chi'}{2r}\right)+\frac{\chi'\eta'}{4}+\frac{\eta'}{2r}, \tag{24}$$

with explicit expressions for ρ and P, given by

$$\rho = \frac{1}{(8\pi+\lambda)(8\pi+2\lambda)r^2}\left[(8\pi+\frac{3\lambda}{2})(1-r\eta'-\eta)+\frac{\lambda}{2}(\eta\chi' r-1+\eta)\right], \tag{25}$$

$$P = \frac{1}{(8\pi+\lambda)(8\pi+2\lambda)r^2}\left[(8\pi+\frac{3\lambda}{2})(\eta\chi' r-1+\eta)+\frac{\lambda}{2}(1-r\eta'-\eta)\right]. \tag{26}$$

The second set of quasi-Einstein equations is given by

$$8\pi\varphi_0^0 = -\frac{h^{*\prime}}{r} - \frac{h^*}{r^2}, \quad (27)$$

$$8\pi\varphi_1^1 = -\frac{h^*}{r}\left(\frac{1}{r} + \chi'\right), \quad (28)$$

$$8\pi\varphi_2^2 = -\frac{h^*}{4}\left(2\chi'' + \chi'^2 + \frac{2\chi'}{r}\right) - \frac{h^{*\prime}}{4}\left(\chi' + \frac{2}{r}\right). \quad (29)$$

with conservation equations

$$P' + (\rho + P)\frac{\chi'}{2} = \frac{\lambda}{8\pi + \lambda}(\rho' - P'), \quad (30)$$

$$(\varphi_1^1)' - \frac{\chi'}{2}(\varphi_0^0 - \varphi_1^1) - \frac{2}{r}(\varphi_2^2 - \varphi_1^1) = 0. \quad (31)$$

Eq.(30) is modified TOV equation in $f(R, T)$ framework.

We describe physical variables in total form as follows

$$\rho^{(tot)} = \rho + \tilde{\beta}\varphi_0^0, \quad (32)$$

$$P_r^{(tot)} = P - \tilde{\beta}\varphi_1^1, \quad (33)$$

$$P_t^{(tot)} = P - \tilde{\beta}\varphi_2^2. \quad (34)$$

Matching conditions

- A space–time representing a compact object is divided by a hypersurface Σ into two distinct regions that are termed as interior and exterior space–times.
- The interior space–time $\mathcal{M}^-$ must match to the corresponding exterior space–time $\mathcal{M}^+$ smoothly at Σ.
- Israel-Darmoise matching conditions.

- The continuity of the first fundamental form:
  $[ds^2]_\Sigma = 0$
  $\Rightarrow (g_{00}^- = g_{00}^+)|_\Sigma$
  $(g_{11}^- = g_{11}^+)|_\Sigma.$
- The continuity of 2nd fundamental form:
  $[K_1^1]_\Sigma = 0$
  $\Rightarrow P_r(R_b) = 0$
  $[K_2^2]_\Sigma = [K_3^3]_\Sigma$
  $\Rightarrow m(\mathcal{R}) = M$

Matching conditions

- The matching conditions are of fundamental importance in the study of stellar objects. The smooth matching of the interior $\mathcal{M}^-$ and the exterior $\mathcal{M}^+$ geometries at the surface of stellar configuration, defined by $\Sigma \equiv r = \mathcal{R}$, is required to ensure a well–behaved compact structure.
- In the present case, the matching conditions are unknown and in principle the Israel-Darmoise matching conditions might not work.
- This is so because the Israel-Darmoise conditions derived in the GR scenario are using the vacuum outer solution i.e., the Schwarzschild solution.

- In $f(R, T)$ theory, we have some extra junction conditions as presented in [J. L. Rosa, Phys. Rev. D 103, (2021) 104069],i.e.,

$$\begin{aligned} [R]_\Sigma &= 0, \qquad [\partial_c R]_\Sigma = 0, \\ [T]_\Sigma &= 0, \qquad [\partial_c T]_\Sigma = 0. \end{aligned} \tag{35}$$

  First two conditions also exists in the framework of $f(R)$ theory, for the models having $f''(R) \neq 0$ [J. M.M. Senovilla, Phys. Rev. D **88**, (2013) 064015], for example the Starobinsky model.

- Extra conditions can be discarded or trivially satisfied via choice of a particular form of the $f(R, T)$ function, Particularly if we choose $f(R, T) = R + \lambda T$.

- In the case of $f(R,T)$ gravity theory, the equivalent solution is not known and its determination depends upon of the shape of the $f(R,T)$ function. Moreover, the presence of the trace T of the energy-momentum tensor would lead in principle to a non-vacuum exterior space-time.
- Moreover, when the MGD or e-MGD are applied, the exterior space-time is receiving contributions coming form the θ-sector, therefore, even in the GR case (in the presence of MGD or E-MGD) the Israel-Darmoise conditions could become invalid.
- So, the junction condition process deserves a thorough and exhaustive study in the context of pure $f(R,T)$ and $f(R,T)$+e-MGD.

Matching conditions

- Let us start by analyzing in general, the possibility of having a vacuum exterior space-time in the $f(R, T)$ context ($\tilde{\beta} = 0$).
  [S.K. Maurya, A. Errehymy, Ksh.N. Singh et al. Phys. Dark Univ. **30** (2020) 100640]
- So the field equations has the form

$$R_{\alpha\beta} = \frac{1}{f_R}\left[T_{\alpha\beta} - (T_{\alpha\beta} + \Theta_{\alpha\beta}) f_T + \frac{1}{2} g_{\alpha\beta} f - (g_{\alpha\beta}\Box - \nabla_\beta\nabla_\alpha) f_R\right].$$

- Next, the $f(R, T)$ function can be seen as the contribution of a purely geometric and matter parts as follows

$$f(R, T) = f_1(R) + f_2(T).$$

- So, taking the vacuum case, that is $T_{\alpha\beta} = 0 (\rightarrow T = 0)$, one obtains

$$R_{\alpha\beta} = \frac{1}{f_{1R}}\left[\frac{1}{2} g_{\alpha\beta}(f_1(R) + f_2(T)) - (g_{\alpha\beta}\Box - \nabla_\beta\nabla_\alpha) f_{1R}\right].$$

Matching conditions

- Thus, it is clear that a vanishing energy-momentum tensor in the framework of $f(R, T)$ gravity theory does not mean a null Ricci tensor like in GR.
- Moreover, $T_{\alpha\beta} = 0$ does not imply $f_2 = 0$, of course this term could contributes with a constant term for example, if $f_2(T)$ function containing an exponential term.
- However, if $T_{\alpha\beta} = 0 \Rightarrow f_2(T) = 0$, the outer manifold is affected by the geometric terms encoded in $f_1(R)$ and f_{1R}.
- This last situation also happens in $f(R)$ gravity theory, however in [B. Whitt, Phys. Lett. B 145, 176 (1984)] it has been shown that in Einsteins frame that Schwarzschild solution is the only static spherically symmetric solution for an action of the form $R + aR^2$
- The simple linear model $f(R, T) = R + \lambda T$ is a good candidate to explore compact structures in order to compare some relevant aspects with GR.

- When the MGD is applied, the exterior space-time receives contributions coming from the φ-sector.
- In this scenario, the exterior region will be represented by the deformed Schwarzschild metric, which is given by

$$\begin{aligned} ds^2 &= -(1-\frac{2\tilde{M}}{r})dt^2 + (1-\frac{2\tilde{M}}{r}+\tilde{\gamma}g^*(r))^{-1}dr^2 \\ &\quad + r^2(d\theta^2 + sin^2\phi d\phi^2), \end{aligned} \tag{36}$$

where

- $g^*(r)$ is the deformation function, related to $\varphi^+_{\alpha\beta}$
- $\tilde{M}$ is the total mass in the exterior region

- From the continuity of 1st and 2nd fundamental forms, we have

$$e^{\chi(R_b)} = 1 - \frac{2\tilde{M}}{R_b}, \quad e^{-\psi(R_b)} = 1 - \frac{2\tilde{M}}{R_b} + \tilde{\gamma}g^*_{R_b} \tag{37}$$

$$(P)^-_{R_b} + \tilde{\beta}\frac{h^*_{R_b}}{8\pi}\left(\frac{1}{{R_b}^2} + \frac{\psi'_{R_b}}{R_b}\right) = \frac{\tilde{\gamma}g^*_{R_b}\left({R_b}^2 - Q^2\right)}{8\pi r^2({R_b}^2 - 2\tilde{M}R_b + Q^2)}. \tag{38}$$

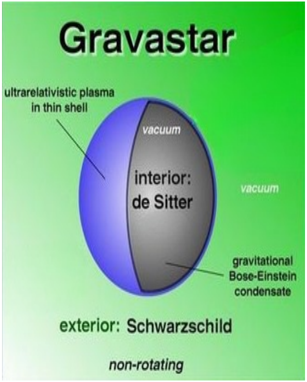
In addition, the decoupling of the Schwarzschild vacuum solution $(T^{+}_{\alpha\beta} = 0)$ and the additional external source $(\varphi^{+}_{\alpha\beta} \neq 0)$ provides the set of equations (27)-(29) for the exterior region $(r > r_{\Sigma} = R_b)$ in the following form

$$8\pi(\varphi^0_0)^+ = -\frac{g^*}{r^2} - \frac{\tilde{\gamma} g^{*'}}{r}, \tag{39}$$

$$8\pi(\varphi^1_1)^+ = -\frac{g^*}{r(r-2\tilde{M})}, \tag{40}$$

$$8\pi(\varphi^2_2)^+ = \frac{\tilde{M}(r-\tilde{M})}{r^2(r-2\tilde{M})^2} g^* - \frac{(r-\tilde{M})}{2r(r-2\tilde{M})} g^{*'}. \tag{41}$$

The gravitational vacuum star (gravastar), proposed by Mazur and Mottola [Mazur, P. O., and Mottola, E.:Report No. LA-UR-01- 5067, arXiv:gr-qc/0109035], is an interesting mathematical model for the description of **an extremely compact stellar structure.**



It consists of three distinct regions:

- $P = -\rho$ (an inner space–time)
- $P = \rho$ (intermediate thinshell)
- $P = 0 = \rho$ (an outer space–time)

- We considered dark energy EoS for the interior of stellar system ($r \leq R_b$), given by

$$P = -\rho. \tag{42}$$

- Using EoS (42) in Eq.(30), we have

$$\rho = \rho_0. \tag{43}$$

where ρ_0 is constant. Thus, Eq.(42) becomes

$$P = -\rho_0. \tag{44}$$

- The gravastar solution with the **undeformed metric functions** $\{\chi,\ \eta\}$ in explicit forms is

$$e^{\chi} = C_2(1 - l^2 r^2), \qquad \eta = 1 - l^2 r^2 \tag{45}$$

where

$$l^2 = \frac{(8\pi + 2\lambda)\rho_0}{3} = \frac{2M}{{R_b}^3} = \frac{R_S}{{R_b}^3}. \tag{46}$$

For the solution of system of equations related to the anisotropic sector, we need to determine $h^*(r)$

- We considered a main characteristic of gravastar structure, i.e., $g_{00} = g_{11}^{-1} = 0$, at the surface $r \rightarrow R_b = R_S$.
- This situation leads to

$$h^*(r) = (1 - l^2 r^2) l^n r^n, \quad \text{with} \quad n \geq 2 \tag{47}$$

- The deformed radial metric component assumes the form as

$$e^{-\psi} = (1 - l^2 r^2)(1 + \tilde{\beta} l^n r^n), \tag{48}$$

with regularity condition

$$\tilde{\beta} \geq -1. \tag{49}$$

The particular expressions for physical variables are

$$\rho^{(tot)} = \frac{3l^2}{8\pi + 2\lambda} + \tilde{\beta}\frac{l^n r^{n-2}}{8\pi}\left[(n+3)l^2 r^2 - n - 1\right], \tag{50}$$

$$P_r^{(tot)} = -\frac{3l^2}{8\pi + 2\lambda} - \tilde{\beta}\frac{l^n r^{n-2}}{8\pi}\left(3l^2 r^2 - 1\right), \tag{51}$$

$$P_t^{(tot)} = -\frac{3l^2}{8\pi + 2\lambda} - \tilde{\beta}\frac{l^n r^{n-2}}{8\pi}\left[(n+3)l^2 r^2 - \frac{n}{2}\right], \tag{52}$$

with the anisotropic factor given by

$$\Delta \equiv P_t^{(tot)} - P_r^{(tot)} = \frac{\tilde{\beta} l^n r^{n-2}}{16\pi}\left(n - 2 - 2nl^2 r^2\right). \tag{53}$$

The Eqs.(50)-(52) describe an anisotropic non-uniform gravastar structure which is ultracompact, with necessary and sufficient conditions for the smooth joining at the boundary

$$1 - \frac{2\tilde{M}}{R_b} + \tilde{\gamma} g^*_{R_b} = 0, \tag{54}$$

$$-\left(\frac{3}{8\pi + 2\lambda} + \frac{2\tilde{\beta}}{8\pi}\right) = \frac{\tilde{\gamma} g^*_{R_b}}{8\pi(R_b - 2\tilde{M})}. \tag{55}$$

.

Here, we have two cases for the exterior region:

1. Standard Schwarzschild solution
2. Deformed Schwarzschild solution

For standard SS, we set $g^*(r) = 0$. then Eq.(55) yields

$$\tilde{\beta} = -\frac{12\pi}{8\pi + 2\lambda}.. \quad (56)$$

For compatibility with regularity condition, we must have

$$\lambda \geq 2\pi(1 - \tilde{\alpha}^2 {R_b}^4). \quad (57)$$

Thus, we have a family of deformed interior solutions corresponding to (57), in such a way that interior MGD gravastar solution and exterior schwarzschild solution smoothly match with each other, for $R_b \to R_S$.

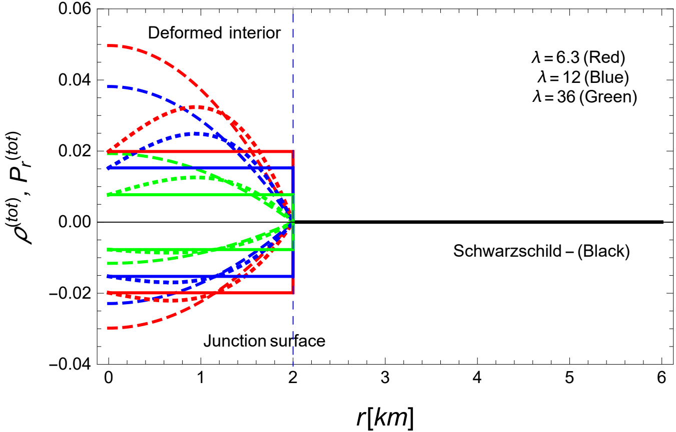


Figure: Variation of the interior total energy density $\rho^{(tot)} > 0$ and interior total radial pressure $P_r^{(tot)} < 0$ for $n = 2$ (Dashed), $n = 3$ (Dotted) and $n >> 2$ (Solid), while exterior solution is represented by the Schwarzschild solution.

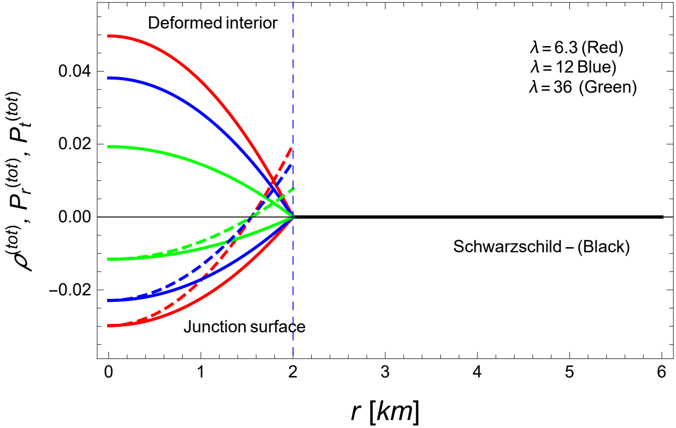


Figure: Variation of the total energy density $\rho^{(tot)}$ (Solid lines, above the x-axis), total radial pressure $P_r^{(tot)}$ (Solid lines, below the x-axis) and total tangential pressure $P_t^{(tot)}$ (Dashed lines), for $n = 2$.

Deformed Schwarzschild solution

- **In this case, the outer space–time receives some contributions from the so-called anisotropic sector, i.e., φ-sector, due to the non-zero $g*$.**
- **Among different choices, we considered an exterior source $\varphi^{+}_{\alpha\beta}$, that satisfies the traceless condition, i.e., $\varphi^{+\alpha}_{\alpha} = 0$.**
- **The exterior deformation function assumes the form**

$$g^*(r) = \frac{1 - 2\tilde{M}/r}{2r - 3\tilde{M}} \ell_c. \tag{58}$$

Thus, we obtained a conformally deformed Schwarzschild solution as follows

$$e^{-\psi} = \left(1 - \frac{2\tilde{M}}{r}\right)\left(1 + \frac{\ell}{2r - 3\tilde{M}}\right). \tag{59}$$

The second fundamental form (55) yields the following expression for ℓ

$$\ell = -8\pi(\frac{3}{8\pi + 2\lambda} + \frac{\tilde{\beta}}{4\pi})\tilde{M}. \tag{60}$$

The above condition together with (59) leads to the result that $e^{-\psi}$ will be positive for all 'r' such that

$$r \quad \geq \quad \left(\frac{3(8\pi + \lambda)}{8\pi + 2\lambda} + \tilde{\beta}\right)\tilde{M}. \tag{61}$$

- **The following expression can be extracted for $\tilde{\beta}$**

$$\tilde{\beta} \quad = \quad \frac{\lambda - 8\pi}{8\pi + 2\lambda}. \tag{62}$$

- **In order to preserve regularity condition in the interior region, we must have**

$$\lambda \geq 0. \tag{63}$$

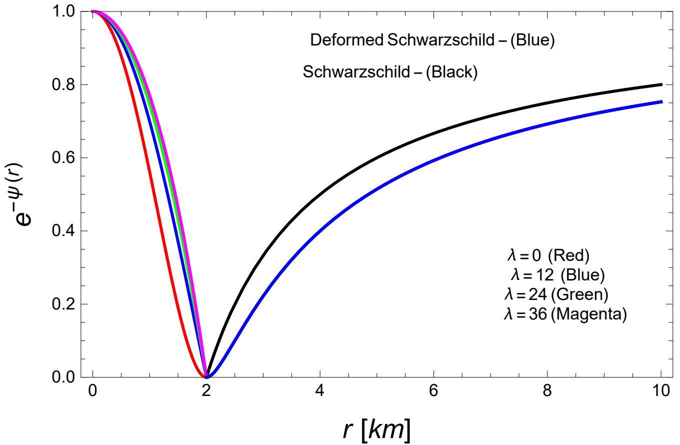


Figure: Evolution of interior as well as exterior radial metric component for the MGD gravastar, with respect to r.

The expressions for the **state parameters in the exterior region** are

$$\rho^{(tot)+} = \tilde{\gamma}({\varphi_0}^0)^+ = -\frac{\ell \tilde{M}}{8\pi(2r - 3\tilde{M})^2 r^2}, \tag{64}$$

$$P_r^{(tot)+} = -\tilde{\gamma}({\varphi_1}^1)^+ = \frac{\ell}{8\pi(2r - 3\tilde{M}) r^2}, \tag{65}$$

$$P_t^{(tot)+} = -\tilde{\gamma}({\varphi_2}^2)^+ = -\frac{\ell(r - \tilde{M})}{8\pi(2r - 3\tilde{M})^2 r^2}, \tag{66}$$

with anisotropic factor

$$\Delta^+ = \frac{\ell(3r - 4\tilde{M})}{8\pi(2r - 3\tilde{M})^2 r^2}. \tag{67}$$

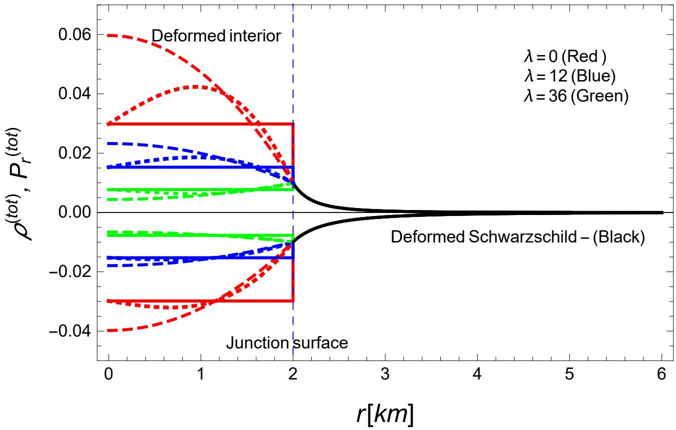


Figure: Variation of the interior total energy density $\rho^{(tot)} > 0$ and interior total radial pressure $P_r^{(tot)} < 0$ for $n = 2$ (Dashed), $n = 3$ (Dotted) and $n >> 2$ (Solid), while exterior solution is represented by deformed Schwarzschild model.

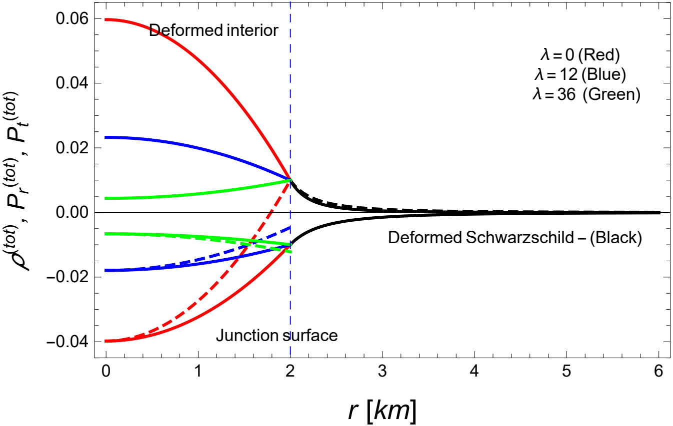


Figure: Variation of the total energy density $\rho^{(tot)}$ (Solid lines, above the x-axis), total radial pressure $P_r^{(tot)}$ (Solid lines, below the x-axis) and total tangential pressure $P_t^{(tot)}$ (Dashed lines), for $n = 2$.

Concluding Remarks

- An anisotropic version of an ultracompact stellar structure of radius $R_S = 2M$, has been developed in the framework of $f(R, T)$ gravity.
- Smoothly joins the standard Schwarzschild solution for all $\lambda \geq 2\pi$.
- Smoothly matches with Deformed Schwarzschild solution for all $\lambda \geq 0$, however, stable solutions have been obtained only for $0 \leq \lambda \leq 8\pi$.


Thank you