Matter-Geometry couplings and the variation of the energy-momentum tensor

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Motivations

- Late-time observations of the universe, suggests that the standard Einstein's theory is insufficient for explaining the universe.
- We have to do something:
  - Adding extra degrees of freedom to the theory; scalar-tensor,...
  - Modifying the way the graviton interact; massive gravity,...
  - Making the geometry richer; Weyl, Cartan, Finsler, higher dimensions,...
  - Modifying the way the matter behaves; $f(R, T, L_m)$, derivative matter,...

Different modified theories of gravity

higher order: $f(R),\ f(G),\dots$

changing the gravitational intractions: dRGT, Weyl/Cartan,...

non-standard matter interactions: $f(T, T_{\mu\nu}T^{\mu\nu})$

$$S = \int d^4x\sqrt{-g}(\kappa^2 R - 2\Lambda) + S_m$$

non-minimal matter-geometry couplings: $f(R, T, R^{\mu\nu}T_{\mu\nu})$

adding extra dof: Galileon, Generaliized Proca,...

higher dimensions: DGP, RS,...

The problem!

- Here, we are interested in theories with $T_{\mu\nu}$ in the action; $f(R,T)$ theories.
- In these theories we have to vary the EM tensor $T_{\mu\nu}$ to obtain the metric field equation.
- The variation can be obtained simply from the definition of the EM tensor:

$$T_{\mu\nu} = L_m g_{\mu\nu} - 2\frac{\delta L_m}{\delta g^{\mu\nu}}.$$

- Then the variation is

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = \frac{1}{2}L_m(g_{\alpha\beta}g_{\mu\nu} - g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) - \frac{1}{2}T_{\alpha\beta}g_{\mu\nu} - 2\frac{\delta^2 L_m}{\delta g^{\alpha\beta}\delta g^{\mu\nu}}.$$

- For a perfect fluid, there is a discussion that since $L_m = -\rho$ or $L_m = P$ does not depend on the metric, then

$$\frac{\delta^2 L_m}{\delta g^{\alpha\beta}\delta g^{\mu\nu}} = 0.$$

The problem!

- Then we obtain

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = \frac{1}{2} L_m (g_{\alpha\beta} g_{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) - \frac{1}{2} T_{\alpha\beta} g_{\mu\nu}.$$

- This argument is suspicious at least in two ways:
  1. L_m should depend on the metric since we have obtain $T_{\mu\nu}$ out of it by varying with respect to the metric.
  2. The above result depends on the form of Lagrangian L_m.
     This is not good since $T_{\mu\nu}$ is independent itself.
- These are our main reasons to search for a better answer...

Thermodynamics considerations

- The first law of Thermodynamics (FLT) and the Gibbs-Duhem (GD) equations are

$$dU = TdS - PdV + \mu dN,$$

$$U = TS - PV + \mu N,$$

- Define:
  the particle number density $n = N/V$,
  entropy per particle $s = S/N$,
  energy density as $\rho = U/V$.
- FLT and GD becomes

$$d\rho = Tnds + \mu' dn, \qquad \rho = \mu' n - P,$$

  where $\mu' = \mu + Ts$ is the Entalphy per particle.
- Differentiate GD and use FLT:

$$dP = nd\mu' - nTds,$$

- In summary: $\rho = \rho(n,s)$ and $P = P(\mu', s)$.
- We have

$$\left(\frac{\partial \rho}{\partial n}\right)_s = \mu' = \frac{\rho + P}{n}.$$

- This implies that $P(\mu', s)$ and $\rho(n,s)$ are Legendre transformation of each other:

$$P(\mu', s) = n\left(\frac{\partial \rho}{\partial n}\right)_s - \rho(n,s).$$

- define:
  the particle number flux density $J^\mu = \sqrt{-g}\, n u^\mu$.
  the Taub current $V_\mu = \mu' u_\mu$, which is the enthalpy flux per particle.
- n can be obtained as
$$n = \sqrt{\frac{g_{\mu\nu} J^\mu J^\nu}{g}}.$$
- With the above definition, one obtains
$$J \equiv \sqrt{-J_\mu J^\mu} = \sqrt{-g}\, n, \qquad J^\mu = J u^\mu,$$
$$V \equiv \sqrt{-V_\mu V^\mu} = \mu', \qquad V_\mu = V u_\mu.$$

- The variation of entropy density s, the ordinary matter number flux density J^{μ}, and the Taub current V_{μ}, wrt the metric tensor vanishes.

$$\frac{\delta s}{\delta g^{\alpha\beta}} = 0, \quad \frac{\delta J^{\mu}}{\delta g^{\alpha\beta}} = 0, \quad \frac{\delta V_{\mu}}{\delta g^{\alpha\beta}} = 0.$$

- The first and second one demand that the entropy production rate and the particle production rate are constant.
- The last one is because we can decompose the Taub current through Pfaff's theorem as

$$V_{\mu} = \varphi_{,\mu} + \alpha\beta_{,\mu} + \theta s_{,\mu},$$

where φ, α, β, θ and s are the velocity potentials (scalar fields) and are independent of the metric tensor.

The story of different Lagrangians

- For a perfect fluid in GR, we have two constraints:
  - particle number conservation: $\nabla_\mu(nu^\mu) = 0$.
  - absence of entropy exchange between two neighboring flow lines: $\nabla_\mu(nsu^\mu) = 0$.
- There are also other constrains:
  - the flow line should be timelike: $u_\mu u^\mu = -1$.
  - boundary conditions for the fluid.

- These constraints can be added to the matter Lagrangian, by Lagrange multipliers

$$S=\int d^4x\sqrt{-g}\left[-\rho(n,s)+J^{\mu}(\nabla_{\mu}\phi+s\nabla_{\mu}\theta+\beta_A\nabla_{\mu}\alpha^A)+\lambda(u_{\mu}u^{\mu}+1)\right],$$

  ϕ, θ, λ and β_A are Lagrange multipliers.
  The α term will take care of the boundary conditions;
  A being the index representing the number of BC.
- We can equivalently imply the constraints to the EOM.
- For λ and β_A we will do that.
- We will see that $-\rho(n,s)\to P(\mu',s)$ will also work.
- For a general theory with matter-geometry couplings:
  we know that the EM tensor is not conserved.
  so, the particle number density and entropy exchange are not necessarily conserved.
  there is no need to add them to the Lagrangian by Lagrange multipliers.

- In order to vary the action, we need to compute variations of all thermodynamics quantities.
- From the definitions of J^{μ} and V_{μ}, one obtains:

$$\delta n = \delta\left(\frac{J}{\sqrt{-g}}\right) = \frac{n}{2}(-g)\,u^{\mu}u^{\nu}\left(\frac{\delta g_{\mu\nu}}{g} - \frac{g_{\mu\nu}}{g^2}\delta g\right) = \frac{n}{2}\left(u_{\mu}u_{\nu} + g_{\mu\nu}\right)\delta g^{\mu\nu}.$$

$$\delta\mu' = \delta V = -\frac{V_{\mu}V_{\nu}}{2V}\delta g^{\mu\nu} = -\frac{1}{2}\mu' u_{\mu}u_{\nu}\delta g^{\mu\nu}.$$

- From the FLT and GD equation we have

$$\delta\rho = \mu'\delta n, \qquad \delta P = n\delta\mu'.$$

which is obtain from the fact that $\delta s = 0$.

- We obtain

$$\frac{\delta\rho}{\delta g^{\mu\nu}} = \frac{1}{2}(\rho + P)(g_{\mu\nu} + u_\mu u_\nu), \qquad \frac{\delta P}{\delta g^{\mu\nu}} = -\frac{1}{2}(\rho + P)u_\mu u_\nu.$$

- It should be noted that from the above equations one obtains

$$V_\mu J^\mu = f(s),$$

where f is an arbitrary function of the entropy per particle s.

- This is the result of the fact that we have assumed conservation of particle production rate.

Variations of the matter sector

- Remembering the definition of EM tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} = L_m g_{\mu\nu} - 2\frac{\delta L_m}{\delta g^{\mu\nu}},$$

  one obtains the EM tensor as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}.$$

- This is true for both Lagrangians $L_m = -\rho$ and $L_m = P$.
- These two Lagrangians are equivalent for a perfect fluid in GR.
- In theories with matter-geometry couplings, the EM tensor can be present in the action, so we should know its variation (at least wrt the metric).

- First note that

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}.$$

- The 4-velocity is defined as $u^\mu = dx^\mu / d\tau$, where

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu,$$

- We have

$$\delta(d\tau) = \frac{1}{2d\tau}\delta(d\tau^2) = -\frac{1}{2}\delta g_{\mu\nu} u^\mu dx^\nu.$$

- The variations of the 4-velocity can then be obtained as

$$\delta u^\mu = \delta\left(\frac{dx^\mu}{d\tau}\right) = -\frac{dx^\mu}{d\tau^2}\delta(d\tau) = \frac{1}{2}u^\mu u^\alpha u^\beta \delta g_{\alpha\beta} = -\frac{1}{2}u^\mu u_\alpha u_\beta \delta g^{\alpha\beta}.$$

$$\delta u_\mu = \delta(g_{\mu\nu} u^\nu) = -\frac{1}{2}(g_{\mu\alpha} u_\beta + g_{\mu\beta} u_\alpha + u_\mu u_\alpha u_\beta)\delta g^{\alpha\beta}.$$

- We can immediately find the second variations

$$\frac{\delta^2 P}{\delta g^{\alpha\beta}\delta g^{\mu\nu}} \equiv \frac{\delta}{\delta g^{\alpha\beta}}\left(\frac{\delta P}{\delta g^{\mu\nu}}\right) = \frac{1}{4}(\rho+P)\big(g_{\mu\beta}u_\alpha u_\nu + g_{\mu\alpha}u_\beta u_\nu + g_{\nu\beta}u_\alpha u_\mu + g_{\nu\alpha}u_\beta u_\mu - g_{\alpha\beta}u_\mu u_\nu + 2u_\mu u_\nu u_\alpha u_\beta\big),$$

and

$$\frac{\delta^2(-\rho)}{\delta g^{\alpha\beta}\delta g^{\mu\nu}} = \frac{\delta^2 P}{\delta g^{\alpha\beta}\delta g^{\mu\nu}} - \frac{1}{4}(\rho+P)(g_{\alpha\beta}g_{\mu\nu} - g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}),$$

- Remember the variation of the EM tensor

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = \frac{1}{2} L_m (g_{\alpha\beta} g_{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) - \frac{1}{2} T_{\alpha\beta} g_{\mu\nu} - 2 \frac{\delta^2 L_m}{\delta g^{\alpha\beta} \delta g^{\mu\nu}}.$$

- Previous works assumed that the last term vanishes for perfect fluid.
- Previous works obtained different results for the above variation for different matter Lagrangians (because of the first term).
- Our calculations give a same result for both Lagrangians:

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = \frac{1}{2} P (g_{\alpha\beta} g_{\mu\nu} - g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) - \frac{1}{2} T_{\alpha\beta} g_{\mu\nu} - 2 \frac{\delta^2 P}{\delta g^{\alpha\beta} \delta g^{\mu\nu}},$$

- This is the final result:

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = \frac{1}{2} P(g_{\nu\beta}g_{\alpha\mu} + g_{\nu\alpha}g_{\beta\mu}) - (\rho + P)u_\mu u_\nu u_\alpha u_\beta$$
$$- \frac{1}{2}\left(T_{\alpha\nu}g_{\mu\beta} + T_{\beta\nu}g_{\mu\alpha} + T_{\alpha\mu}g_{\nu\beta} + T_{\beta\mu}g_{\nu\alpha} - T_{\mu\nu}g_{\alpha\beta} + T_{\alpha\beta}g_{\mu\nu}\right).$$

- Independent of the choice of matter Lagrangian: $L_m = -\rho$ or $L_m = P$.

Variation of the EM tensor

- Define a new tensor

$$\bar{T}_{\mu\nu} = (\rho + P)u_\mu u_\nu + \frac{1}{2}P g_{\mu\nu}.$$

with $\bar{T} = -\rho + P$.

- The variation can be written as

$$\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = -\frac{1}{2}\left(\bar{T}_{\beta\nu}g_{\mu\alpha} + \bar{T}_{\alpha\nu}g_{\mu\beta} + \bar{T}_{\alpha\mu}g_{\nu\beta} + \bar{T}_{\beta\mu}g_{\nu\alpha} - \bar{T}_{\mu\nu}g_{\alpha\beta} + \bar{T}_{\alpha\beta}g_{\mu\nu}\right) - (\rho + P)u_\mu u_\nu u_\alpha u_\beta.$$

When to use equation of state $P = P(\rho)$

- We should imply the equation of state after the variation; on EOM.
- Suppose we have an EOS of the form $P = \alpha\rho^n$.
- The variations with respect to $-\rho$ and P result in a perfect fluid EM tensor.
- However, varying wrt $\alpha\rho^n$ (implying the EOS to the action) gives something wrong.

- In $f(R,T)$ gravities, these quantities are well-known

$$\mathbb{T}_{\alpha\beta} \equiv g^{\mu\nu}\frac{\delta T_{\mu\nu}}{\delta g^{\alpha\beta}} = -3(\rho+P)u_\alpha u_\beta - \frac{1}{2}(\rho+3P)g_{\alpha\beta}. \qquad \text{(For both Lagrangians)}$$

- Also, we have

$$\frac{\delta T}{\delta g^{\alpha\beta}} = T_{\alpha\beta} + \mathbb{T}_{\alpha\beta}.$$

- The trace

$$\mathbb{T} \equiv g^{\mu\nu}\mathbb{T}_{\mu\nu} = -T = (\rho - 3P).$$

- In the comoving frame, we have

$$\mathbb{T}^{\mu}_{\nu} = \frac{1}{2}\mathrm{diag}\left(5\rho+3P, -\rho-3P, -\rho-3P, -\rho-3P\right).$$

A simple comparison

- Remember, for the present calculation

$$\mathbb{T} \equiv g^{\mu\nu}\mathbb{T}_{\mu\nu} = (\rho - 3P).$$

- For previous calculations

$$\mathbb{T}_{\mu\nu} = (L_m - 2P)g_{\mu\nu} - 2(\rho + P)u_\mu u_\nu. \qquad \text{(Lagrangian dependent)}$$

For $L_m = -\rho$, we obtain

$$\mathbb{T} \approx -2(\rho + 3P).$$

For $L_m = P$, we obtain

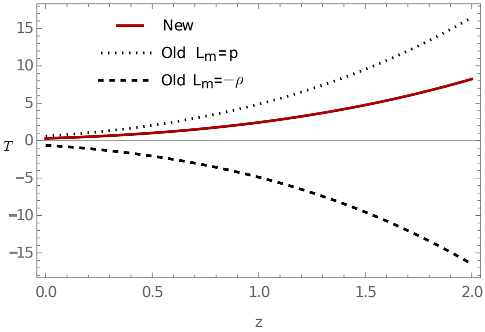
$$\mathbb{T} \approx 2(\rho - P).$$

- Assuming an FRW universe with

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right),$$

and assuming conserved matter source with

$$\rho_m = \frac{\Omega_{m0}}{a^3}, \quad \rho_r = \frac{\Omega_{r0}}{a^4},$$



- Consider a simple model

$$S=\int d^4x\sqrt{-g}(\kappa^2R+f(R,T)+L_m).$$

- Assume the matter source to be a prefect fluid with $L_m=-\rho$ or $L_m=p$.
  No difference for the new calculations, but we get different result with previous one.
- The equation of motion can be obtained as

$$\kappa^2G_{\mu\nu}-\frac{1}{2}f\,g_{\mu\nu}+f_RR_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R=\frac{1}{2}T_{\mu\nu}-f_TT_{\mu\nu}-f_T\mathbb{T}_{\mu\nu}$$

- The conservation equation is

$$\left(\frac{1}{2}-f_T\right)\nabla^\mu T_{\mu\nu}=\left(T_{\mu\nu}+\mathbb{T}_{\mu\nu}\right)\nabla^\mu f_T+f_T\left(\nabla^\mu\mathbb{T}_{\mu\nu}+\frac{1}{2}\nabla_\nu T\right).$$

- The only difference with previous calculations is in the tensor $\mathbb{T}_{\alpha\beta}$.
  present result:

$$\mathbb{T}_{\alpha\beta} = P g_{\alpha\beta} - 2T_{\alpha\beta} - 2g^{\mu\nu}\frac{\delta^2 P}{\delta g^{\alpha\beta}\delta g^{\mu\nu}}. \qquad \text{(Lagrangian independent)}$$

  previous result:

$$\mathbb{T}_{\alpha\beta} = L_m g_{\alpha\beta} - 2T_{\alpha\beta}. \qquad \text{(Lagrangian dependent)}$$

- For a perfect fluid, we have

$$g^{\mu\nu}\frac{\delta^2 P}{\delta g^{\alpha\beta}\delta g^{\mu\nu}} = \frac{1}{4}(\rho + P)(2u_\alpha u_\beta + g_{\alpha\beta}).$$

- Two calculations are equivalent only if $P = -\rho$ (not an ordinary matter EOS).

- For brevity, assume a very simple case $f(R,T) = \alpha|T|^n$.
- The EOM can be written as

$$\kappa^2 G_{\mu\nu} = \frac{1}{2}T_{\mu\nu} + \frac{1}{2}\alpha|T|^n g_{\mu\nu} + n\alpha|T|^{n-1}(T_{\mu\nu} + \mathbb{T}_{\mu\nu}).$$

- We will assume that the universe has dust with $P = 0$.
- We have $T = -\rho$.
- Also:
  present calculation:

$$\mathbb{T}_{\mu\nu} = -3\rho u_\mu u_\nu - \frac{1}{2}\rho g_{\mu\nu}.$$

  previous calculations:

$$\mathbb{T}_{\mu\nu} = \begin{cases} -2\rho u_\mu u_\nu - \rho g_{\mu\nu}, & \text{for } L_m = -\rho, \\ -2\rho u_\mu u_\nu, & \text{for } L_m = P, \end{cases}.$$

- Transforming to dimensionless variables

$$\tau = H_0 t, \quad H = H_0 h, \quad \bar{\rho} = \frac{\rho}{6\kappa^2 H_0^2}, \quad \beta = (6\kappa^2 H_0^2)^{n-1}\alpha.$$

- The cosmological equations can then be obtained as

$$h^2 = \bar{\rho}_m - \beta(3n+1)\bar{\rho}_m^n, \qquad h' = -\frac{3}{2}\left(\bar{\rho}_m - 4\beta n\bar{\rho}_m^n\right).$$

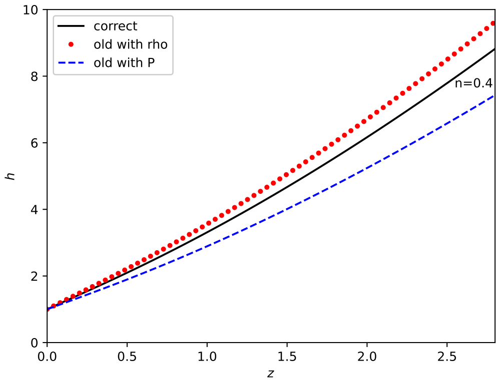
- Previous calculations leads to:
  for $L_m = -\rho$

$$h^2 = \bar{\rho}_m - 2\beta\bar{\rho}_m^n, \qquad h' = -\frac{3}{2}\left(\bar{\rho}_m - 2\beta n\bar{\rho}_m^n\right).$$

for $L_m = P$

$$h^2 = \bar{\rho}_m - 2\beta(2n+1)\bar{\rho}_m^n, \qquad h' = -\frac{3}{2}\left(\bar{\rho}_m - 2\beta n\bar{\rho}_m^n\right).$$

- They predict different results.
- For $n = 0.4$, $\Omega_{m0} = 0.3$ we obtain



- Let us solve the correct theory.
- Transforming to z coordinates defined as

$$1 + z = \frac{1}{a}.$$

- From the relation $h(z=0) = 1$ one obtains

$$\beta = -\frac{1 - \Omega_{m0}}{(1+3n)\Omega_{m0}^{n}}.$$

- Now, use the Likelihood analysis for the observational data on the Hubble parameter in the redshift range $z \in (0.07, 2.36)$.
- The Likelihood function

$$L = L_0 e^{-\chi^2/2},$$

where L_0 is the normalization constant

- The χ^2 function

$$\chi^2 = \sum_i \left(\frac{O_i - T_i}{\sigma_i} \right)^2 .$$

i counts the data points,
O_i are the observational value,
T_i are the theoretical values,
σ_i are the errors associated with the ith data obtained from observations.

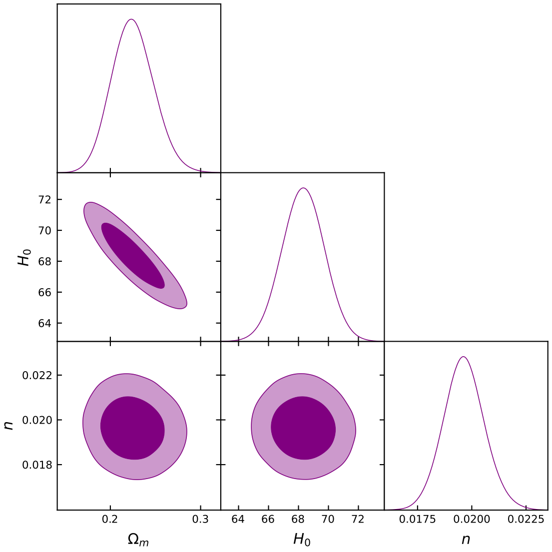
- The best fit values of the parameters n, Ω_{m0} and H_0 at 1σ confidence level, can be obtained as

$$\Omega_{m0} = 0.224^{+0.024}_{-0.023},$$
$$H_0 = 68.396^{+1.401}_{-1.408},$$
$$n = 0.018^{+0.001}_{-0.001}.$$

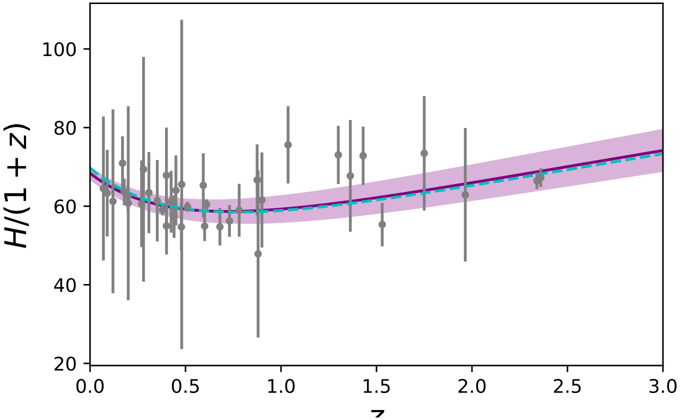
- The best fit value for β is

$$\beta = -0.756^{+0.031}_{-0.030}.$$

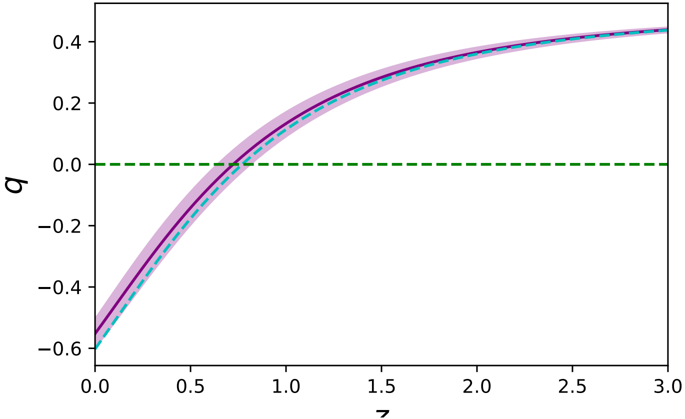
The corner plot



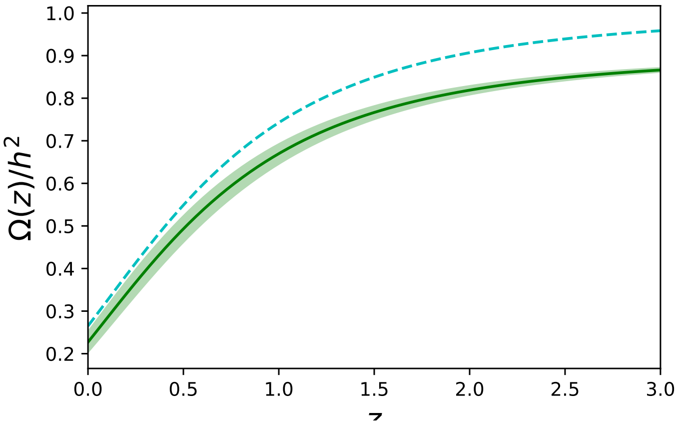
The Hubble parameter with GR plot and observational data



The deceleration parameter together with GR plot



The matter abundance together with GR plot



- We have suggested a new calculation for obtaining the variation of the EM tensor.
- This is not the most general calculation, but it guarantees the independence of the variation from the matter Lagrangian.
- The new calculation gives a same variation for both Lagrangians.
- Theories with matter-geometry couplings will be affected by this calculation.

Thanks for your attention!

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